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ABSTRACT

This eight part guide was developed to assist individuals in improving mathematics education in Georgia schools, particularly in planning curriculum at the local level. The guide was prepared on the basis of successful teaching practices and recommendations from recognized educators. Information regarding considerations unique to teaching the middle grade learner is given. In addition, issues in mathematics education today such as how to teach problem-solving, how to use technology, and how to overcome mathematics anxiety are addressed. Also included are suggestions for ways teachers can plan student activities that help middle grade learners acquire the appropriate mathematical concepts and skills. The guide is divided into five major sections: (1) developing curriculum; (2) setting instructional goals/objectives; (3) planning instruction; (4) evaluating mathematics learning; and (5) a bibliography of instructional resources. Provided in additional sections are: appendices (containing a statement on organization for the essentials of education and a National Council of Supervisors of Mathematics position paper on basic mathematics skills); a glossary; and suggested career awareness activities. (JN)

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Mathematics for Georgia Middle Grades

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Georgia Department of Education

Mathematics for Georgia Middle Grades

Division of Curriculum Services
Office of Instructional Services
Georgia Department of Education
Atlanta, Georgia 30334
Charles McDaniel
State Superintendent of Schools
1982

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Marjorie P. Economopoulos, Fulton County Schools, Atlanta
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Foreword

Mathematics education is reviewed continually at every level to keep instruction responsive and current. Resulting changes in mathematics education are based on developments in theories of learning, findings regarding instructional practices in the classroom, public opinion and most important, the evolving process of mathematics itself.

The Georgia Department of Education appointed a committee to review and revise the mathematics curriculum guide for the middle grades. This guide, *Mathematics for Georgia Middle Grades*, is the result of the committee's study and writing.

We appreciate the time and effort the committee members gave to this project and commend them for this excellent publication. We believe that as educators in middle schools throughout Georgia use this guide in their curriculum planning, the mathematics education in those schools will improve significantly.

Charles McDaniel
State Superintendent of Schools

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Introduction

The purpose of this guide, *Mathematics for Georgia Middle Grades*, is to assist those who are concerned with improving mathematics education in Georgia schools. Information provided here is intended to help in planning curriculum at the local level.

The writers have prepared this guide on the basis of successful teaching practices and recommendations from recognized educators. Information regarding considerations unique to teaching the middle grade learner is given. Issues in mathematics education today such as how to teach problem solving, how to use technology and how to overcome mathematics anxiety are addressed. The writers have included suggestions for ways teachers can plan student activities that help middle grade learners acquire the appropriate concepts and skills.

We hope that this guide will help local planners as they review, revise or develop the mathematics curriculum and thereby improve mathematics instruction for all students in Georgia schools.

Lucille G. Jordan
Associate State Superintendent
Office of Instructional Services

R. Scott Bradshaw, Director
Division of Curriculum Services

Developing a Middle Grades Mathematics Curriculum

Developing a Middle Grades Mathematics Curriculum

A major goal of schools is to prepare students to function in modern society—both at the present time and in the future. In order to plan towards this goal the school system or the school should prepare a curriculum for each subject area. Such plans should be developed by groups representing various sectors within the school system and the community. This curriculum guide, *Mathematics for Georgia Middle Grades*, was designed to assist those involved in planning a local system, local school or individual classroom mathematics curriculum.

It is assumed that a local overall curriculum planning committee with members representing all subject areas has been established and has given direction and input to the leaders developing the mathematics curriculum. The mathematics curriculum committee (or individual teacher) planning or revising a mathematics curriculum should develop steps or tasks to follow and designate people to be responsible for the tasks and associated time frames.

A curriculum cannot be developed quickly. Adequate time must be given for study and discussion of (a) trends and issues in the field of mathematics, (b) ways students learn and (c) strategies for teaching.

Suggested steps for developing a mathematics curriculum for middle grades are presented on the following pages. Of course these tasks may need to be adapted to accommodate the local situation. Time frames are not included since time needed for these tasks will vary from school system to school system.

Steps for Developing A Middle Grades Mathematics Curriculum

Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
1. Formulate Mathematics Curriculum Committee composed of • local curriculum director • local mathematics supervisor or designated leader • representatives from mathematics teachers in elementary, middle and secondary schools • media specialist • representatives from other curriculum areas (to be called on as needs arise) • guidance counselor (to be involved periodically) • representatives from the community (to be called on periodically)	Curriculum leaders, general and mathematics	
2. Develop goals of mathematics learning. • Study general and mathematics goals of local and state educational agencies and state and national professional mathematics organizations. • Study (or formulate) philosophy of the school system regarding general education and mathematics education. • Consider local student needs, present and future.	Mathematics Curriculum Committee	Nature of Middle Grades Learner, Modeling and Problem Solving, Goal Setting, Strands and objectives, Processes, Personalizing Instruction, Special Consideration, Support Systems Evaluation, Appendices, Resources
3. Study materials and provide sufficient time to discuss. • How do students learn mathematics?	Subcommittee of Mathematics Curriculum Committee	Nature of Middle Grades Learner, Modeling and Problem Solving, Goal Setting Strands and Objectives, Processes, How to Begin, Personalizing Instruction,

Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
How do students learn mathematics as a language? As a science? As a collection of skills? As an art?		Special Considerations, Support Systems, Evaluation, Appendices, Resources
• What are the students' attitudes towards mathematics? What changes in attitudes and appreciations do you wish a modified mathematics program to attain?		
• Are there specific mathematical needs for your community? Are there particular needs in careers typically pursued by your students?		
• Does present curriculum include sufficient opportunities for problem solving and evaluation of problem solving?		
• What strategies of teaching should be employed? Are a variety of strategies used in teaching?		
• What major topics of mathematics should be addressed in the curriculum? At which grade levels or within which units of study should these be addressed? Note: *Keep notes on readings to help in writing the guide—especially activities and references.* These findings should provide a framework within which the curriculum can be built.		

	Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
4.	Review the Statewide Criterion-Referenced Test (CRT) objectives and high school graduation requirements (both state and local).	Mathematics Curriculum Committee	Strands and Objectives, Appendices
5.	Develop student objectives for mathematics education and indicate those skills which lead to the expected skills of graduating seniors whether they enter the world of work or postsecondary schools. The objectives should be coded to the state CRT objectives and the Basic Skills Test indicator clusters.	Mathematics Curriculum Committee	Modeling and Problem Solving, Goal Setting, Strands and Objectives, Processes, Appendices, Resources
6.	Review the existing curriculum to ascertain if the skills expected of secondary school students are included and are at appropriate grade levels in the middle grades to ensure opportunities for students to develop these skills; indicate those missing from curriculum and place those desired in the list of objectives.	Members of Mathematics Curriculum Committee	
7.	Review existing curriculum in terms of stated goals, objectives and local student needs; indicate inconsistencies and include those objectives desired into the revised list of objectives.	Members of Mathematics Curriculum Committee	
8.	Develop a format for the curriculum guide.	Members of Mathematics Curriculum Committee	Appendices

Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
9. List topics to be studied in the middle grades; indicate the grade level(s) in which they will be placed. Of course some topics will be included in each grade—introduced, developed and reinforced—as needed for the spiral approach to learning. These topics may be the same as the strands listed in this guide—or they may be subtopics within the strands or a combination of subtopics from more than one strand, e.g., standard geometric shapes and measuring area. Decisions on scope and sequence of topics should be carefully planned based on such considerations as (a) mathematical skills needed prerequisite to the study of a given topic, (b) the movement of students from one class or school to another, (c) scheduling the use of available resources (both materials and people). Of course flexibility must be allowed so as to take advantage of opportunities which may present themselves.	Subcommittees and the whole of the Mathematics Curriculum Committee	Nature of Middle Grades Learner, Modeling and Problem Solving, Processes, Personalizing Instruction, Special Considerations, Sample Activities, Appendices, Resources
10. Identify, select or write student activities to develop each topic and to help students attain the objectives. Code the activities to the topics and the objectives. Teachers should plan co-operatively so that repetition from one grade level to another is used for reinforcement rather than in haphazard situations where it is boring and useless to students.	Subcommittee of Mathematics Curriculum Committee	Modeling and Problem Solving, Processes, Personalizing Instruction, Special Considerations, Support Systems, Sample Activities, Evaluation, Appendices, Resources, Careers in Mathematics.

Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
11. Identify, select or develop materials needed for the activities. Identify or select other resources helpful to students in achieving the objective.	Subcommittees and the whole Mathematics Curriculum Committee, additional mathematics teachers and consultants	Resources, Careers In Mathematics.
12. Review the offerings developed through number 10 and respond to the following questions: • Have appropriate offerings been provided for all levels of students? • Have a sufficient quantity of activities been planned to offer students or to offer teachers samples so as (a) to assist them toward attaining minimum requirements for high school, (b) to provide a wide variety of activities beyond the minimum, (c) to provide for a variety of teaching and learning styles. • Are objectives stated so that evaluation of student attainment can be measured? • Are there provisions for a variety of strategies including discovery approach, small group or individual activities, observation, exploration, investigation, inquiry, organization of ideas, organization of data, applications to other disciplines and reinforcement? • Is the level of student involvement as high as possible? • Are the activities appropriate with respect to student needs, abilities and interests?	Subcommittees and the whole Mathematics Curriculum Committee	

Task	Responsibility of	Resources in Mathematics for Georgia Middle Schools
• Based on present inventory are all needed materials on hand? If not, list missing material ranked from most to least needed.		
13. Revise the plans based on answers to task 11 above.	Mathematics Curriculum Committee	
14. Identify, select or develop procedures for evaluating the objectives—for evaluating both the program and individual student progress.		
15. Develop a plan to field-test the program.	Mathematics Curriculum Committee	
16. Select schools and teachers to field-test the program.	Mathematics Curriculum Committee, supervisors and administrators	
17. Field-test the program—keep notes regarding changes needed in the program.	Designated teachers and supervisors	
18. Review/revise curriculum—use questions in previous steps to develop plan for review/revision.	Mathematics Curriculum Committee	
19. Plan for evaluation of mathematics curriculum.	Mathematics Curriculum Committee	Evaluation
20. Formulate and implement staff development plan.	Mathematics Curriculum Committee, supervisors and administrators	
21. Implement the mathematics curriculum plan.	Mathematics teachers, administrators and supervisors	
22. Evaluate the mathematics curriculum each year.	Mathematics Curriculum Committee, teachers, administrators and supervisors	Evaluation
23. Review findings of evaluation each year and plan revision where needed.	Mathematics Curriculum Committee, teachers and supervisors	

Setting Goals and Objectives

SETTING GOALS AND OBJECTIVES

The Middle Grades Learner
Modeling and Problem Solving
Goal Setting
Strands and Objectives
Processes

Setting Goals and Objectives

In determining goals and objectives at the state, system, building or classroom level, one of the first steps must be a consideration of the philosophical bases of the program. Beliefs and perceptions about effective learning will greatly influence the nature of the students' experiences in a particular school system, school, and/or classroom. If educators are to grow in effectiveness, it is important that they examine their basic philosophies about mathematics education in the middle grades. These perceptions will naturally include some beliefs about the learners themselves, the mathematics to which the learners will be exposed and techniques for combining these two sets of beliefs in an effective program.

The first section on the nature of the middle grades learner discusses the particular characteristics of this age student and includes some suggestions for using these characteristics to enhance the experiences of students rather than let them serve as deterrents to learning. Problem solving as the major emphasis in any mathematics program is the theme of the second section. Curriculum decisions should not be made until some of the issues addressed in this section are considered. A section on goals is then included to present additional considerations in setting goals. Once the philosophical bases have been determined and the goals set, the objectives can then be formulated. A section on objectives is included which contains an explanation of the role of state objectives and local objectives and a list of statewide objectives for the middle grades. The last section deals with the role of processes in the middle grades mathematics program. As with other parts of the guide, the order of presentation of the sections does not necessarily represent the order in which they should be studied. The topics in the sections are interdependent and therefore should be studied as a whole rather than as separate topics.

The Middle Grades Learner

Who are the middle grades learners? What are they like? They are happy one minute and sad the next. They are responsible one day and irresponsible the next. One minute they want to tell you what to do and the next minute they want you to tell them every move to make. They also vacillate from being logical to irrational, endearing to exasperating, verbose to nonverbal or withdrawn, serious to silly, naive to sophisticated, adult-like to childish and so on. Their physical growth also occurs in spurts and this erratic growth often results in awkwardness, sporadic energy, fatigue and restlessness. These constantly changing emotional, intellectual and physical growth factors are confusing to these students and cause them to constantly reassess themselves. Further, middle grade years, when youngsters are approximately 10 through 14 years of age, are critical in the development of self-concept, a very important contributor to both personal and academic success.

The adults who work with these students have difficulty understanding these erratic behaviors. The knowledge and subsequent understanding of these traits are essential for the effectiveness of any adult working with middle grades students. Such knowledge and understanding should not, however, lead merely to an acceptance and tolerance of these characteristics. To adopt a tolerant, and basically a neutral, position rather than a positive one could result in misuse, or even loss, of at least four critical years of instruction. Students will experience physical growth during this period of time. They should experience intellectual growth as well.

Although there is some consistency in each student and/or group of students, planning mathematics instruction for a class of diverse individuals who will likely be different each day is not an easy task. Even planning for one unpredictable and changing middle grades learner is difficult. Although programs will vary greatly, a common goal of each should be that the program be developed cooperatively by individuals who are knowledgeable of and sensitive to the unique situation of these students in the middle grades.

The individual characteristics of the middle grades learner have many possible ramifications for the mathematics program. It would not be possible in this section to deal fully with the characteristics and how they individually or collectively affect given educational situations. Not only are the characteristics numerous, but they are also complex since there are some characteristics which might conflict when implementation strategies are considered. For example, ability grouping enables teachers to deal more effectively with the diverse academic levels present in a middle grades program. Ability grouping, however, often causes severe self-concept problems at this particular age when the development of self-esteem is so difficult for students. Another example of solutions that assist in one area but might create problems in another area is the use of flexible scheduling and flexible groupings. These changing schedules and grouping patterns are excellent for providing a variety of mathematical experiences but do not provide the stability for those students who need a very secure and structured environment. The complexity of such learning situations does not mean there are no solutions. They certainly can be and should be addressed on an ongoing basis at the system, school and classroom level. Obviously, compromises and tradeoffs will have to be made and continuous evaluation will be essential in planning these learning situations.

Most of the solutions or decisions concerning the mathematics program for middle grades students must be made at the local level. This section of the guide will contain some suggestions for eliminating certain aspects of mathematics programs that might be counterproductive to the overall growth of these students. Suggested ways to capitalize on some of the characteristics and use them to enhance the intellectual growth of these students in mathematics will also be presented.

Self-concept

Certainly one of the most critical needs of students who are 10 to 14 years of age is the need to feel good about themselves. As they reach the age where they begin to look at themselves and reflect on who they are and who they would like to be, they are often disappointed in themselves. This disappointment of course, manifests itself in many ways—from the student's becoming extremely withdrawn and antisocial to becoming obnoxiously loud and cocky to cover up feelings of inadequacy. Almost everything that is done in the schools during this time will affect this characteristic either positively or negatively. In general, schools need to provide an environment that assures students the best possible opportunity to develop self-esteem. There are four areas that seem to cause more difficulty than others in developing self esteem in middle grades students—homework, testing, excessive competitive aspects of the mathematics program and expectations.

Homework

Certainly there is great support for a return to more homework as a method of reinforcing basic skills in mathematics. There are, however, certain cautions that might be mentioned in connection with middle grade learners. Since their problem with authority often is more home-centered than school-centered, and since many parents have difficulty themselves with mathematics, homework often creates problems in the home which result in setting the student back rather than helping him/her to move forward. Several suggestions that might be made in the area of homework are to

- assign homework as a **follow-up** activity, not before a student has a fairly clear understanding of the concept. This approach might require multiple homework assignments for different readiness levels in the class. These assignments might be completely different in content or maybe in quantities of different portions within the assignments. Twenty sentences of one type might be necessary for one student where five may be sufficient for another.
- communicate with parents concerning their role in the homework process, such as providing specific time and place for study as opposed to giving specific help with content.
- communicate with other teachers on long-range assignments and effective ways of dealing with specific students in the area of homework.
- provide some choices in homework assignments.
- make some group assignments where two or three students are responsible for certain tasks as a group. Students will often fulfill their responsibility to each other when they will not assume it for adults. This suggestion might also encourage students to interact and discuss mathematics. This process should be monitored carefully, however, to assure that one or two students do not do all the work.

Needless to say, there are many possibilities, and no one technique will work for all students all of the time. Varying the format of homework and reducing the possibility of creating problems at home should help in lessening the negative aspects of homework for middle grades learners. Homework should enhance the school program, not destroy the gains which have been made. If ways cannot be found for homework to be helpful, perhaps there is justification for it being eliminated.

Testing

Probably the single most significant factor contributing to the hostility against mathematics and low self-esteem in mathematics is testing. As with homework, testing certainly cannot and should not be eliminated since it is needed for diagnosing and planning for each student, program planning and feedback to parents and students. A major concern regarding testing this age student is communication. For example, there may be a communication problem involved in the following quiz.

Add the following and **simplify** your answers.

X 1) $\begin{array}{r} 3\frac{2}{5} \\ +2\frac{4}{5} \\ \hline 5\frac{6}{5} \end{array}$

2) $\begin{array}{r} 2\frac{1}{3} \\ +3\frac{1}{3} \\ \hline 5\frac{2}{3} \end{array}$

X 3) $\begin{array}{r} 4\frac{2}{5} = 4\frac{8}{20} \\ +2\frac{3}{4} = 2\frac{15}{20} \\ \hline 6\frac{23}{20} \end{array}$

X 4) $\begin{array}{r} 5\frac{1}{2} = 5\frac{5}{10} \\ +3\frac{4}{5} = 3\frac{8}{10} \\ \hline 8\frac{13}{10} \end{array}$

X 5) $\begin{array}{r} 1\frac{3}{4} = 1\frac{21}{28} \\ +2\frac{3}{7} = 2\frac{12}{28} \\ \hline 3\frac{34}{28} \end{array}$

What is the student communicating to the teacher on this test?

"I don't know how to add fractions."

"I don't know what simplify means."

"I don't know **how** to **simplify** my answers."

"I don't like to simplify my answers, and I hope you won't mark it wrong."

"I'm not going to simplify my answers. That's dumb, and I don't care if you mark it wrong. I'll probably fail math anyway."

What is the teacher communicating to the student by giving a grade of 20% on this quiz?

"You don't know how to add fractions."

"You don't know how to follow directions, and you're not going to pass this course until you do."

"You don't know how to simplify your answers."

What was the teacher testing and consequently trying to communicate to the student? Was it adding fractions, simplifying fractions or following directions? If all three are being tested, is the teacher communicating to the students that they have, in fact, learned to add fractions but not the other two tasks?

In addition to the communications issue, the dependence on tests, with their threatening aspects, as the only source of motivation probably turns more middle grades students against mathematics than any other factor. The frequency of tests in mathematics is often a problem. Giving tests once a week means the practice of (1) reviewing for the test on one day, (2) testing on the next day and (3) returning the tests on the third day. This process will result in some negativism toward mathematics on at least three of the five days in a week.

A thorough discussion of assessment and its implications for the middle grades is included in the section on evaluation in this guide.

Excessive Competition

Although competition is normal and certainly basic to the American way of life, most middle grades authorities agree that competition should be reduced drastically during this developmental period. They often recommend the reduction or elimination of social pressures such as dances and other formalized boy-girl relationships. Also discouraged are competitive sports which reward the physically mature, do not provide positive experiences for the majority of the students and can actually be detrimental to students' self-concepts. Likewise, authorities suggest eliminating, or at least reducing, excessive competitiveness in the area of intellectual growth. The most obvious considerations that affect this area are those associated with organizing for instruction. The section of the guide that deals with personalizing instruction addresses this concern. Other areas that might be considered are *required* participation in mathematics fairs, contests or similar events. Beta Club and other secondary school reward systems might add to the already complex problem of developing self-esteem for this age group. Authorities cite the incidence of suicide, the second leading cause of death for this age, as evidence of the inability of these students to deal with pressures in the areas of social, physical and intellectual development. Decisions in the mathematics program should be made with staff input and be based upon research findings. Since these students live in such a competitive society, it is often difficult to convince some educators, parents and students that middle grades students already have an abundance of pressure that is self-induced and that cannot be controlled. Consequently, the schools should be as sensitive as possible in program planning to provide an environment that is supportive of all children in mathematics, not just the socially, physically and intellectually *mature* student.

Not only are there major programmatic concerns in mathematics which need to be considered in the area of competition, but there are also everyday classroom activities which are very important. Each classroom teacher must strive to provide a classroom environment where students feel free to take chances in mathematical problem solving, to guess and possibly be wrong, and to try ideas and find they might not produce results. Since these students often ridicule each other, mathematics teachers must assume the responsibility of eliminating as much of this overt and damaging behavior as possible. A positive atmosphere can often be accomplished by having students' discussion through the use of brainstorming procedures that facilitate participation by all students. Growth in mathematics for middle grades students, particularily in problem solving or thinking skills, will occur much more readily in a supportive atmosphere where students work together and not against one another. Teachers should be very careful about comparisons of physical appearance, e.g., height, weight or other sensitive areas when choosing relevant examples to use with students.

Expectations

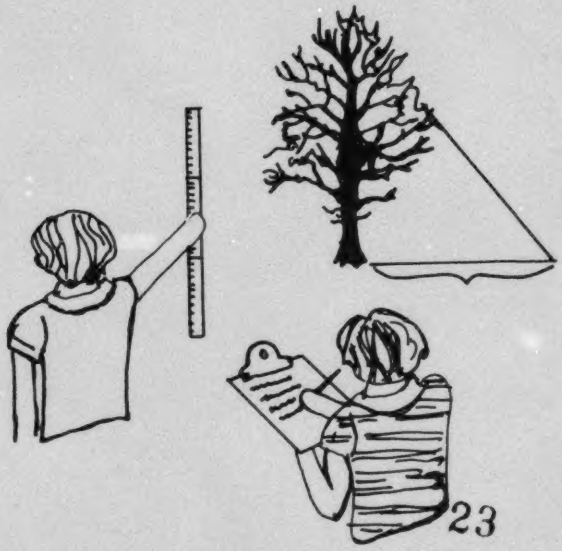
Very few teachers or school administrators have not participated in a parent conference and heard a comment like, "I don't really **expect** Johnny to do well in math. He got that honestly from me. I could never do well in math." Teachers are keenly aware of the negative effect of the expectations of some parents. Professional educators sometimes communicate this same kind of low expectation and do so in a variety of ways. It may be exhibited in the way teachers organize for instruction (See the section *Personalizing Instruction*). It may also be reflected in the assumptions some teachers have that students low in computational skills cannot successfully participate in problem solving ventures. Another example is the practice of asking higher level questions to only the talented students and then probing for the answer while accepting, "I don't know," from a student for whom the teacher's expectations are low. Many subtle remarks are made to middle grades students by school personnel which reflect low expectations. Since it is so difficult to change the self-concept of these students, it is extremely important that the teacher communicate positive expectations.

These four areas have been given as examples of programmatic concerns that should be addressed by every teacher and considered in the development of each mathematics program because of the possible negative impact on middle grades students. A more positive approach in considering the ramifications of the characteristics of the middle grades learner on programs would be to identify certain characteristics of the learner and to use these characteristics to benefit the learner in mathematics. The following five examples might be used by local system groups in taking this approach.

Using Characteristics Positively

The students' need for developing their own value systems

Since students are seeking to develop their own value systems, pursuing this topic might be an excellent opportunity to work on positive attitudes towards mathematics. Many children have accepted the attitudes of their parents toward mathematics and consequently may have fears and a dislike of mathematics. There are specific suggestions regarding this topic in this guide in the article entitled *Mathematics Anxiety*.



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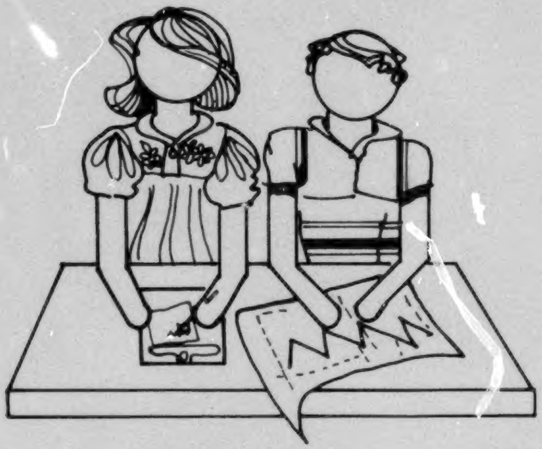
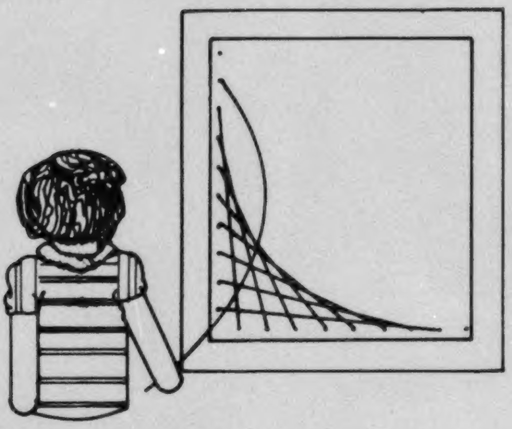
The students' extreme need for peer interaction and approval

There are many ways to utilize this characteristic of students to work for, rather than against, the learning of mathematical concepts. One of the most obvious is peer tutoring. Peer tutoring not only benefits the student having difficulty but also the student providing assistance. Verbalizing processes increases the understanding by the student who is giving the explanation. It also helps to develop in the tutor a positive self-concept which is so difficult for students to maintain at this age. The student being tutored can benefit not only because of the additional help they are provided but also because they are often more willing to listen to a peer tutor. Needless to say, there can be problems with anyone other than a trained professional being used in the instructional program; therefore, the process should be closely monitored and evaluated. Also, the teacher should be cautious about misusing peer tutoring. The bright student should not be overused in this way and consequently not be provided more stimulating enrichment activities.



The students' interest in a variety of topics

Students' preoccupations with areas other than mathematics need not hinder their interest in mathematics. For example, a student who is extremely interested in art will likely become very interested in geometry taught via geoboards, string art, paper folding, symmetrical designs, tiling or tesselations. Atheletes will give almost all tasks in statistics their undivided attention if the data are related to sports. In addition to capitalizing on existing interests, these youngsters are often willing to explore new areas of interest.



Rapid physical development manifested by rapid growth spurts

Learning centers and other similarly structured activities in the classroom are readily accepted by students who sit for many hours a day. Also, the outdoor mathematics emphasis could change this characteristic to a positive contributing factor to the mathematical growth of middle grades students. There are several sections of this guide which should provide assistance in this approach. In particular, see "Ratios, Proportions, and the Great Outdoors" in the "Activities" section of the guide.

The extreme idealism and need for heroes and adult models

There are many heroes in the field of mathematics, such as Pascal, Gauss, Pythagoras, Descartes and Noether, with very interesting biographies that should appeal to students and stimulate their interest in mathematics. These heroes might be presented in mini-units as enrichment, as a *lead-in* to a specific unit associated with the mathematician or as a research effort. There are also heroes in the community that could be called upon to assist with the mathematics program. If the student/parent relationship is rather positive, the use of a parent can often create a positive attitude for a student in mathematics. Using high school students who provide good models will fulfill this need of middle grades students and greatly enhance the program.

In summary, educators should study not only the mathematics curriculum for middle grade students but also the learners themselves. Planning mathematics programs in local school systems and individual schools should utilize this knowledge. Certain aspects of the program such as homework, testing, competition and teacher expectations, tend to present substantial problems unless there is deliberate planning in the areas. Finally, many of the characteristics of middle grades youngsters can be used in a positive way to enhance growth in mathematics rather than impede the growth. The challenge for the mathematics teacher in the middle grades is tremendous. The remainder of the guide will attempt to provide assistance to teachers in meeting this challenge of facilitating growth of the students in these critical years.

Modeling and Problem Solving

Learning to solve problems is probably the most important single process of anyone's education. People in every walk of life are faced with problems to solve. Many of these problems are routine and may actually not be problems at all. Sometimes it is not easy to determine whether a situation is a problem for a particular individual. What is a problem to one person may be merely an exercise to another. If a task requires only routine application of a procedure, it is not really a problem.

Most problems that people encounter in their lives do not come neatly written as textbook type *word problems*. They are usually encountered as problem situations which need to be clarified into problem statements, then approached and solved.

Middle grades learners should have experiences in solving problems that are already formulated and should also have experiences in problem seeking, problem finding and problem formulation.

Mathematical modeling and problem solving seem to go hand in hand. A model is a representation of a real-world situation just as a map of a city (the model) is a representation of the streets of that city (the reality). Learning mathematics can be enhanced by understanding the global uses of concepts as models for phenomena in the world, and problem solving makes direct use of mathematical models. A simple example of mathematics encountered in the middle grades will help illustrate this point. Students learn to manipulate a formula such as $a = bc$ into $b = a/c$ and $c = a/b$. Specific examples illustrate the generality and adaptability of such multi-purpose models and in fact the power of such a simple statement. The same basic mathematical model can represent the relationship between distance, rate and time ($d = rt$) or it can represent Ohm's Law ($I = E/R$). Dr. Richard Skemp (1971, 1979), a noted educational psychologist who specializes in the theory of learning mathematics, makes a case for teaching for such higher levels of abstraction. On the other hand, in order to solve problems, students need to find a mathematical model that fairly well fits with the problem situation.

In *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (National Council of Teachers of Mathematics, 1980), the first recommendation is that "problem solving be the focus of school mathematics in the 1980s. Included in the discussion of problem solving is the statement, "Problem solving involves applying mathematics to the real world, serving the theory and practice of current and emerging sciences, and resolving issues that extend the frontiers of the mathematical sciences themselves."

Suggested actions for the recommendation include the following.

> The mathematics curriculum should be organized around problem solving.
>
> The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, processes, and modes of presentation that encompass the full potential of mathematical applications.
>
> Mathematics teachers should create classroom environments in which problem solving can flourish.
>
> Appropriate curricular materials to teach problem solving should be developed for all grade levels.
>
> Mathematics programs of the 1980s should involve students in problem solving by presenting applications at all grade levels.

The middle grade learners should have experiences solving routine word problems as preparation for and in conjunction with their experiences of formulating problems to solve more real-world problems. George Polya, through his book *How to Solve It* (1957), can assist the instructor in teaching problem solving strategies and help students become aware of the strategies and processes they are using and can draw upon for future use. He separates the problem solving processes into four basic steps.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back and checking the results

Expanding on Polya's work to include real-world problems that are not clearly formulated, one might go through steps such as these.

1. Formulate a problem statement
2. Analyze the problem (understand)
3. Model the problem (devise a plan)
4. Solve the problem (carry out the plan)
5. Evaluate the solution (looking back and checking)

The first step in **Problem Formulation** includes the ability to ask questions or pose problems whose answers can shed some light on the characteristics of the problem situation not yet defined. Students need experience in deciding which aspects of a situation are important enough to include in a problem statement. Often brain-storming techniques are useful in determining considerations upon which to focus. Sometimes there are several important considerations and constraints of a problem situation. Students must then order their considerations by importance and be willing to make trade-offs.

Analysis of the problem includes understanding the problem statement. It can involve restating the problem in different words, identifying what is known, i.e., relevant information or data (or realizing the need to collect such information) and identifying what is not known and needs to be known. Polya's strategies are useful in analyzing problems at this stage.

> You have to **understand** the problem. **What is the unknown? What are the data? What is the condition?** Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
>
> Draw a figure. Introduce suitable notation.
>
> Separate the various parts of the condition. Can you write them down?

Students can be taught strategies for understanding a problem.

Modeling the problem will include translating the information found in the analysis into appropriate mathematical symbols. This step is important, for the conditions of the problem are made explicit and must be correct. A mathematical model can take the form of sentences in mathematical language, a graph, a table or chart, a formula, a diagram, a geometric figure or a combination of these and other formulations.

A mathematical model must be complete enough to describe the problem fairly well and simple enough to solve. A perfect model for a real-world situation usually does not exist. One hopes for a rather good model. In general, a model should

- include as many of the main characteristics of the given situation as practical,
- be designed so that these characteristics of the situation are related in the model as they are in the original real situation or problem statement and
- be simple enough so that the mathematical problems associated with the model can be solved.

The sophistication of the model will depend in part on the mathematical awarenesses of the students and the technology available, e.g., one would not want to use a model where calculus is needed to solve the mathematics problems for middle grade students or to manipulate hundreds of numbers in a data pool without a computer or at least calculators available.

Some of Polya's advice is sound for the search for appropriate mathematical models.

> Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a **plan** of the solution.
>
> Have you seen it before? Or have you seen the same problem in a slightly different form?
>
> **Do you know a related problem?**
>
> **Look at the unknown!** And try to think of a familiar problem having the same or a similar unknown.
>
> **Here is a problem related to yours and solved before. Could you use it?**
>
> . . .

Solving the problem is essentially carrying out the plan. Once a mathematical model has been chosen, solving the explicit mathematical problem(s) is next. In Polya's words, "Carrying out your plan of the solution, **check each step.** Can you see clearly that the step is correct? Can you prove that it is correct?

Skills students need in order to solve the mathematical problems are those emphasized in the objectives in this guide.

Evaluating the solution includes Polya's looking back strategies.

> Examine the solution obtained. Can you **check the result?** Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method for some other problem?

In addition to checking that the solution to the mathematical problem fits that problem, a broader evaluation must occur. Students should look back to the original real-world situation to see if the solution is adequate in this less clear context. It may be that this evaluation will lead to reformulation of the problem statement since in fact the solution is not adequate or that the model needs revision.

It should be emphasized at this point that the progression through the problem solving strategies is not necessarily linear and that some back and forth movement may be necessary. For example, once a problem statement has been formulated, the process of analyzing and understanding the problem may lead back to revision of the original problem statement. Such movement from one step to another will be common and necessary as understanding of the problem situation progresses and is refined.

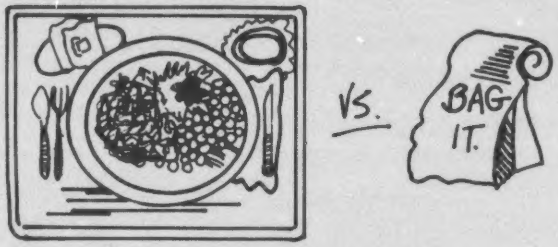
An example of a problem situation that is manageable for middle grades students will help clarify some of the strategies and processes involved.

Problem Situation: Is it better to buy lunch at school or bring a lunch from home?

Formulate a problem statement. Questions concerning food could arise in many varied contexts including an outgrowth of a nutrition unit or as part of a discussion following a polling by students where they asked others whether they brought their lunches to school.

The initial discussion of the problem can lead students to various interpretations of the word *better* in the original problem situation: Is it **better** to buy lunch at school or bring a lunch from home? Several students may consider cost, others may bring up taste and still others may want to focus on the nutrition aspect of school versus home-made lunches. After much debate and possibly some arguing (with the teacher keeping it friendly), students may decide that cost of the respective lunches is their primary interest. At this point, the teacher can encourage students to tell which they think costs more—school lunches or those prepared at home. Various opinions will be given.

Is it better to buy lunch at school or bring a lunch from home?



The next part of the problem situation that needs clarification is exactly **what** do students bring from home in their lunches. The cost of a school lunch will be fixed and obvious. The students will need to go through a similar brainstorming process to define a *typical* lunch brought from home. They may even conduct a quick survey of the students in their own class who bring lunch or may consider a more extensive information collection, such as a sample of the school population who bring lunches.

Once the students clarify the vague terms in the original problem situation, they will need to decide on a clear formulation of the problem statement. They will make mistakes and may end up in the problem analysis stage before they actually formulate an adequate problem statement. This is possibly the most difficult time for the teacher who wants to guide the students and help them do it right the first time. Allowing the students to make their own *mistakes* is very hard for teachers, but at the same time very important for students. An atmosphere that allows for risk-taking and makes it all right to be imperfect will help students become more creative problem solvers. After all, in an open-ended situation of problem formulation, there are many right answers and students must feel free to try to find them.

Eventually students may reach a tentative problem statement such as, Is it **cheaper** to buy lunch at school or bring a typical lunch from home? where typical lunch is specified precisely.

Analyze the problem. The analysis will lead to a more thorough understanding of the problem statement and may even lead students to modify their problem statement. Considerations of cost may require further clarification brainstorming activities. Students may consider a lunch from home to be free since their parents are providing it. They may come to decide to include their parents' costs for this analysis. Which costs are to be included?

Questions that may arise from the students (or be asked by the teacher) include the following.

Do we include only the cost of items in the lunch itself?

Do we want to include hidden costs such as electricity for the refrigerator, lights in the kitchen, preparation time, etc.?

How do we find out the cost of a slice of bread or a piece of sandwich meat or an apple?

Students may decide to ignore hidden costs and focus on actual cost of the parts of the lunch. Questions about the cost of the school prepared lunch may be minimal, but a student may bring up the fact that school lunches are often subsidized.

Decisions must be made to further clarify the problem statement even in the analysis stage. Once these are settled, modeling the problem can begin.

Model the problem. Assume the students have decided that among the typical items in a school lunch brought from home are two slices of bread and a piece of sandwich meat. The students may decide that the cost of two slices of bread is the proportion of these two slices to the total number of slices in a loaf times the cost of one loaf of bread. This is the mathematical model.

$$\frac{\text{2 slices of bread}}{\text{total slices in loaf}} \times \text{cost of loaf} = \text{cost of 2 slices.}$$

Similarly for sandwich meat

$$\frac{\text{1 slice of meat}}{\text{total slices in package}} \times \text{cost of package} = \text{cost of 1 slice of meat.}$$

Even now in the modeling stage a decision must be made on white bread versus whole wheat and brands of both bread and sandwich meat. This decision is part of the problem formulation stage. The teacher must be patient and let students realize the need for such specifics and point out to the students how they need to fluctuate from one step to another and back again.

Similar mathematical models can be made for the other items in the well-defined typical school lunch. At this point in the process, the teacher will realize that the effort so far has been worth it. The students are **experiencing problem solving** in its true sense.

Solve the problem. This stage is probably the easiest for both the students and the teacher. The students have the mathematical problems set up and can proceed to find the solutions. The total cost of the bought lunch is the summing of all the parts. Comparisons of the cost of a school bought lunch and the typical one brought from home can be made.

Evaluate the solution. Does the solution fit the initial conditions of the problem? Does it fit with the students' early guesses and feelings? What would have been the consequences of making different decisions related to the typical lunch? Or to the clarifying of the word better in the original problem situation to include nutrition or something else? Have we answered the original problem? During the evaluation process, the teacher can point out the processes used in solving this problem including the cycling back that was often necessary.

The teacher can play an active role in helping students feel good about themselves during the entire process. For example, eliminating references to particular individual's ideas throughout the brainstorming activities is important. Collecting many ideas on the blackboard or overhead projector and reserving judgment temporarily of any of the ideas will help students see the array of possible decision paths and avoid embarrassment of individual students. Later decisions need to be made regarding which ideas to keep. With a large variety of responses on the board or overhead, **individuals** are not singled out to be disregarded, rather the **ideas** are chosen that will lead the students down one of the possible paths of formulating a problem statement and solving it.

More assistance for the teacher in teaching problem solving will be found throughout this guide. Sections with specific suggestions include "Planning Instruction: Special Considerations—Reading for Problem Solving" (with particular emphasis on reading mathematics and solving word problems) and "Setting Goals and Objectives: Processes" (helps clarify processes students and teachers use) and "Sample Activity—Personal Characteristics" which includes some ill-defined problems for students to clarify. Teachers with students in the upper middle grades may want to consult the section on problem solving in the curriculum guide *Mathematics for Georgia Secondary Schools* (1981) which has a detailed discussion of problem situations and many specific examples.

The challenge to teach problem solving has been made. Teachers will begin to meet this challenge and find that they and their students find mathematics more meaningful through useful applications. To be able to solve problems is, after all, a major goal of all education, and students and their teachers will find the achievement of solving a **real** problem to be very satisfying.

Goal Setting

In organizing for mathematics instruction at the middle grades level, the professional staff of the school system or school must take into account the needs of the community and the particular students being served, along with the educational goals and the resources within the community and the schools. Plans at the school system, school and classroom levels are needed as outlined in the section *Developing a Middle Grades Curriculum*. There should be much study and discussion, as suggested in this section, regarding learning mathematics, attitudes towards mathematics, specific needs for mathematics in the community and students' future careers involving mathematics. The curriculum planners should then come to an agreement as to their expectations. They must look at the performance of their students to determine if the present goals and offerings for the students are appropriate to those expectations. They should review the high school graduation requirements of the state and local school boards in relation to those expectations. The curriculum planners then should accept their present goals or revise them.

The goals should reflect the set of processes the planners have accepted or identified from the study of the section on processes found in this guide. The goals should also reflect the concepts and skills the planners expect of the students who complete the middle grades. Concepts and skills may be identified through the use of the objectives for the statewide criterion-referenced tests and of those objectives stated in this guide. Of course more specific objectives may be needed at the system and school levels. These decisions must be made prior to planning the specifics of evaluating instruction — both for the program and individual students—and, of course, prior to planning the instructional activities.

Immediate goals should also be identified by using school system and school reports of the statewide criterion-referenced tests and other evaluation results to identify those objectives needing special attention. Local objectives should then be identified which correlate to these criterion-referenced objectives needing special attention.

Teachers and other school staff members must be continuously mindful of applications of mathematics in today's world. They should also be cognizant of changes caused by rapidly advancing technology. Many of these changes are found as new applications of mathematics as well as tools used in the applications.

The statement of goals should include the expectation that students be able to use tools such as calculators and microprocessors in solving problems, including applications of mathematics. Instructional activities should then include opportunities for students to use these tools in problem solving situations.

Individual teachers of the middle grades must have copies of the goals and objectives in mathematics for the school system and the school and of the processes, concepts, and skills which need specific emphasis as indicated through assessment. The teacher should sort objectives as to those *needed to know* and *nice to know*.

Using evaluations from previous teachers, student test reports and observations in the classroom, the teacher should identify the needs and strengths of individual students. Those students needing particular help because of severe learning problems or advanced levels of learning should be identified. These students should be referred to appropriate staff members for help — directly to the student or indirectly through the teacher.

After goals are set both for groups within the classroom and for individual students, then plans for assessment and for instructional activities can begin.

Strands and Objectives

Once goals have been established, specific objectives can be determined. This guide uses *Strands* to help teachers of mathematics organize for instruction. As strands or fibers are twisted together to form a rope, the strands or central areas of study are twisted together to form a mathematics curriculum. There are six explicit strands in the guide. They are (1) Sets, Numbers and Numeration, (2) Operations, Their Properties and Number Theory, (3) Relations and Functions, (4) Geometry, (5) Measurement and (6) Probability and Statistics.

There are no specific strands called Problem Solving or Computation. Even more than the other strands, these are considered to be a part of all mathematics and are to be considered in a cross-strand approach. Computation is basic to mathematics and is a necessary skill in each strand. Problem solving is even more inclusive and a crucial area of concern for teachers and students. A global goal for instruction is that students be able to **solve problems** that exist today and those that will exist in the future. Training in how to attack a problem situation is of the utmost importance. See the section "Modeling and Problem Solving" for a more extensive discussion of this topic.

The ordering of the strands in this guide does **not** imply the order of presentation of subject matter; that is, one strand need not be completed or even begun before proceeding to another. Further, the organization of topics into strands does not imply that classroom activities need to include those areas of mathematics that appear in a single strand. In fact, many activities can and should cut across strands. See the "Sample Activities" in this guide.

For each of the six strands there is a description of the strand and a list of objectives. During the middle grades, the learner should be encouraged to construct knowledge in order to master the various objectives. Mastery, however, is not expected to occur during the first introduction of a topic or concept. The material is to be introduced and expanded as the student proceeds through the middle grades. This **spiral** approach allows for the development of knowledge to become a continuing process and an expansion of understanding as mathematical thoughts are revisited and refined.

The collections of objectives in the strands should be viewed as units. Therefore, the order of the objectives does not necessarily imply the sequence of instruction. Further, the volume of material in the "Sample Activities" for particular strands or objectives does not imply that one topic is more important than another. Student activities are presented as examples of how areas of mathematics may be introduced and reinforced. The sample activities support the theory that learning is experiencing.

Each of the objectives that follow may be viewed as a global objective for the overall middle grades mathematics program. Particular systems, schools or teachers may write more specific objectives to meet individual needs. School systems may also wish to write specific objectives for each grade level. Further, the placing of an objective in a particular strand, while done with a great deal of forethought, is in some sense arbitrary. Strands are simply a way of organizing the various objectives. The objectives are to be viewed as a whole. A specific example of where a distinction between strands is difficult is for the measurement and geometry strands. Objectives which may be considered metric geometry are in the measurement strand since introduction of various geometrical concepts such as volume of a box, area of a rectangle or perimeter of closed curves are usually introduced by measurement experimentation and teaching for discovery learning. For this reason, these objectives have been placed in the measurement strand. Using a cross-strand approach to teaching eliminates potential problems associated with the placement of objectives into strands.

Processes that students use to learn and solve problems are considered to be an important part of the mathematics curriculum. Understanding the processes involved in problem solving will assist students to become better problem solvers. Teachers need to become more aware of the processes both they and their students use and how understanding these processes can facilitate learning. A discussion of processes and processing follows the objectives for each strand.

Sets, Numbers and Numeration

Introduction

The concept of a set is a useful tool in the study of mathematics, and the language of sets enables one to communicate mathematical ideas with clarity and precision. In the middle grades many different kinds of sets are studied, for instance, sets of points in geometry, sets of equivalent fractions in the development of the rational number concept and sets of factors in number theory. Activities should be selected which will help pupils develop understandings and skills necessary to identify, describe and classify sets.

Whole numbers may be defined as properties of finite sets; more precisely, a whole number is an abstract concept associated with a class of equivalent (finite) sets. For instance, the number 5 is the common property of all sets which can be put into one-to-one correspondence with the set of fingers on one hand. Counting is the process of assigning a whole number to a finite set. In the middle grades the concepts of whole number and numeration are extended from the foundations for understanding number and for learning the system of symbols for denoting numbers, to an emphasis on the place value principle used in writing numerals. Some experiences should be provided for students to work with numbers that are extremely small or extremely large. The national budget and other numbers used in social studies, as well as many numbers used in science lessons provide examples to use to illustrate a need for this skill. These large numbers will also provide opportunities for discussing rounding off numbers. Students will be able to see that a million dollar project does not cost exactly one million dollars.

Certain applications of whole numbers lead to the important concept of **ordered pairs of numbers.** For instance, the whole numbers 2 and 5 are components of the ordered pairs symbolized in the following examples.

(a) 2/5 (read "2 for 5"), where the ordered pair expresses the rate of 2 balloons for 5 cents.

(b) 2/5 (read "2, 5"), where the ordered pair expresses the date, February 5th.

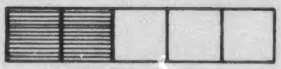
(c) (2,5) (read "2, 5"), where the ordered pair is a member of the solution set for the open sentence $\square + 3 = \triangle$.

(d) (2,5) (read "2, 5"), where the ordered pair is associated with a point in the coordinate plane.

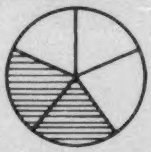
(e) 2/5 (read "2 to 5"), where the ordered pair represents the ratio of the number of holidays to the number of school days in a particular week.

All of the above situations involving ordered pairs of whole numbers should be dealt with in the middle grades. It should be noted that the symbol for an ordered pair has meaning only in terms of the context in which the ordered pair is used.

The ordered pair (2,5) is studied in still another context in middle grades mathematics, the *fraction context.* In that case, 5 is the count of parts into which a unit or a set of some kind has been partitioned, and 2 is the count of those parts which have been singled out or marked for attention, as in the following illustrations.



(2 parts out of 5 parts in a unit strip)



(2 parts out of 5 parts in a unit disc)

(2 balls out of 5 balls in a unit set)

The symbol for the number pair used in the fraction context is generally written as 2/5 and read "2 over 5" or "2 fifths."

Use of the term *rational number* for the number pair in this case should be postponed until pupils reach a level of mathematical maturity sufficient to understand that a rational number is an equivalence class. For instance, the infinite set of ordered pairs,

$$\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \frac{10}{25}, \frac{12}{30}, \ldots$$

represents one (exactly one) rational number.

There are several numeral forms which the school pupil learns to use in expressing fractions (or rational numbers).

(1) The fraction (or ratio-like) form, e.g., 13/5 (read "13 over 5" or "13 fifths").

(2) The decimal numeral form, e.g., 2.6 (read "2 and 6 tenths" or "26 tenths").

(3) The mixed numeral form, e.g., 2 3/5 (read "2 and 3 fifths").

It should be noted that what one has often called *decimal numbers* or *mixed numbers* are, in fact, fractions (or rational numbers) expressed in decimal numerals or in mixed numerals. It is extremely important for pupils to know that the symbols 13/5, 2.6, 2 3/5 all represent exactly the same number and that a preference for one of the numerals depends, generally, on the application or use one makes of the number.

In addition to the study of the whole numbers and rational numbers, there is another kind of number which is an important part of the developing concept of number, that is, the integers. As with many of the topics in the guide, there is no particular time or place in the curriculum when the study of integers should be initiated or completed. Certain informal uses of negative numbers or readiness experiences with signed numbers can occur as early as the primary grades. Many pupils have experiences with temperatures above and below zero and with gains and losses in football yardage statistics or other game-related scores. In working with a numberline some pupils may wonder about the numbers on the other side of zero. Such experiences or ideas should be built on and expanded throughout the grades.

Sets, Numbers and Numeration

Objectives

The student should be able to do the following.

1. Use the language of sets to describe and organize information
2. Make and interpret generalized statements using *and, or, all, some* and *none*
3. Read and write large numbers
4. Use expanded exponential form to demonstrate place value of the decimal system
5. Name the pair of numbers associated with fractional parts of (a) units, (b) sets
6. Discriminate between a pair of numbers used in a rate context and a pair of numbers used in a fraction context
7. Identify the set of equivalent fractions associated with a given point on a number line
8. Use fractions to represent rational numbers
9. Use decimals to represent rational numbers
10. Order any two or more given rational numbers (whole numbers, decimals, fractions and mixed numbers)
11. Recognize different representations of the same number and determine if two number representations are equivalent, e.g., 2/5 = 40% = 0.4 = 0.40 = 4/10; 2 + 7 = 4 + 5 = 9 = 18/2 = 900%; 1.7 = 1 7/10 = 170/100 = 70%; .34 = 3.4×10^{-1}
12. Change one number representation to another representation, e.g., given 40%, change to 0.4; given 2/5, generate some equivalent fractions 4/10, 6/15, 12/30, 40/100; change 1/3 to 0.3; and $-(-2) = 2$
13. Apply an appropriate number representation to a particular situation, e.g., the discount of a $400 item with 25% off can be found by 1/4 × 400 or 0.25 × 400
14. Determine approximations of numbers by rounding off, such as to the nearest thousand, hundred, tenth or hundredth, e.g., 1532.17 is approximately 1500 to the nearest hundred and approximately 1532.2 to the nearest tenth; and a calculator print out of 4.367215 is approximately 4 to the nearest unit or 4.4 to the nearest tenth
15. Recognize and use directed numbers (integers) in describing everday situations
16. Construct the set of integers, i.e., the union of the set of whole numbers and the set of their opposites
17. Order any two or more given integers

Operations, Their Properties and Number Theory

Introduction

The purpose of this strand is two-fold. One is to build the concept of operations and their properties, and the other is to develop interest in number relationships through number theory.

In building the concept of operations and their properties, it is important to distinguish between operations and computations. An operation is a particular association of a certain member of a set to a given pair of numbers of the set. Computation is the manipulation of numerals to determine the number that results from combining two numbers by means of an operation.

Operations and their properties in the guide are studied in terms of their meanings. The student is introduced to each of the four fundamental operations of arithmetic through some physical situation. The initial interpretation of the operation is derived from the physical situation.

The concept of number operations evolves from two main physical sources — one is the number associated with sets of discrete objects, and the other is the measurement of continuous quantities. Therefore, the activities in this strand are concerned with sets (discrete) and the number line (continuous).

In using the operations, the student must know which is applicable to the situation in the problem at hand. For example, the number pair (6, 2) can be associated with 4, with 8, with 3, or with 12. The students must select the appropriate operation for solution of their problem situation, and they must know which number is associated with the operation.

The four fundamental operations with integers or rationals cannot always be introduced to the pupil using physical situations. Therefore, separate activities are needed to introduce the student to an interpretation of these operations for these sets of numbers. As with whole-numbers, writing symbols for the operations is more effectively understood by the student after generalizations have been firmly grasped. For example, to be able to write ${}^{+}3 + {}^{-}5 = {}^{-}2$ with understanding the students must have experience with interpreting a physical situation such as: if one goes east three miles (${}^{+}3$) and then west five miles (${}^{-}5$), ones location is then two miles west of the place one started (${}^{-}2$).

After acquiring a basic understanding of operations in a number system, the pupil may use this knowledge to explore number ideas through number theory. In working with operations one begins with a pair of numbers to which a single number is assigned by a specific operation; in studying number theory one encounters such experiences as looking *inside* a single number and studying the relationship between numbers of a particular set. For example, one may look closely at the single number 49 to find answers to questions as: Is it a prime number? Is it an odd number? Is it a square number? The student may also investigate number patterns in order to recognize numerous relationships of numbers, for example, extending patterns, skip counting classifying numbers as odd or even, prime or composite, and many other topics included in typical modern middle grades mathematics textbooks.

Investigating numbers and number patterns provides more challenging and appealing activities for a student to use in learning mathematical concepts and basic facts than the traditional drill activities or practice exercises. The study of number theory is especially interesting in that a solution to one problem often becomes the basis for another problem.

In some modern textbooks number theory is treated as a separate topic. In others the concepts are included under topics such as multiplication of whole numbers. It is important for the teacher to see that concepts of number theory should be taught as a foundation for other concepts. For example, the study of least common multiple would be necessary before addition of certain rational numbers.

In the early grades the students should learn about properties of operations by manipulating objects and observing the number relationships on which the properties are based. It is not as important for them to know the names of the properties as it is for them to apply them when appropriate. In the middle grades the pupil should be able to identify the properties by name.

Operations, Their Properties and Number Theory

Objectives

The student should be able to do the following.

1. Select the arithmetic operation(s) appropriate to a given physical situation or illustrate a given arithmetic operation by a physical situation
2. Identify odd and even numbers
3. Generalize results of operations with odd and even numbers
4. Identify prime and composite numbers
5. Determine the factors of any whole number
6. Determine the greatest common factor of a set of numbers
7. Determine the least common multiple for a set of numbers
8. Identify and continue number patterns
9. Demonstrate immediate verbal recall of any basic facts
10. Recognize and apply properties such as commutative, associative and distributive
11. Compute efficiently—both with and without a calculator—using whole numbers, fractions, decimals and negative numbers
12. Use estimation to solve a problem situation if an approximate answer is adequate or to check the reasonableness of the results of a calculation
13. Use properties of (a) additive identity and additive inverses and (b) multiplicative identity and multiplicative inverses

Relations and Functions

Introduction

Relations, the idea of pairing or corresponding in a certain order, is basic to all mathematics. Beginning even at the preschool level, the pupils can learn through experience to recognize relations, to use them in formulating their own ideas and to show in communicating to others that they are developing intuitively a pattern of organized thinking in nonnumerical situations. By using relational thought patterns in their early, experiences, they establish readiness for extending these concepts in mathematical situations as they encounter them in their development. Therefore, the teacher should see that from the beginning a foundation for correct concepts is laid so that unlearning will not be necessary later.

Pupils encounter many nonnumerical relations; many can be found in stories for children. Some of these nonnumerical relations, such as *belongs to*, *is brother of* and *is in the same house as* should be used before numerical relations to illustrate the meaning of relations. These relations can also be used to lead into numerical relations since they can be examples of correspondences of *one-to-one*, *many-to-one*, *one-to-many* or *many-to-many*. Such correspondences are basic to the idea of number, to the relations *equal to*, *less than* and *more than*, and to the operations with numbers. If these concepts are not fully developed prior to the middle grades, some attention should be given to them before considering other mathematical relations.

There are several special kinds of relations with special names. One of these, called an equivalence relation, is associated with the process of classification. Classification is the process of partitioning a set of elements into different subsets in which no element can belong to more than one subset. This, too, can be introduced through nonnumerical situations. For example, a set of blocks can be separated into classes on the basis of color provided the colors are distinct. Or, a set of coins can be partitioned into subsets according to value. Such subsets, or classes, are called equivalence classes, and the relation exemplified by their membership, *same color as* or *same value as*, is called an equivalence relation. When school children are classified by grade in school, if no pupil can be in more than one grade, the different grades represent equivalence classes, and the equivalence relation is *is in the same grade as*. Equivalence relations are very important in mathematics. The most familiar is the one called *is equal to*, but many others are encountered as the pupil progresses through mathematics.

Another special kind of relation is that known as a function, or mapping. Although the concept of a function is one of the unifying themes of mathematics, it is unwise to introduce pupils to the concept by giving a formal definition. If the pupils have sufficient practice in pairing elements of one set with elements of another while studying relations in general, those having the special property required of functions will not be difficult to identify.

It is also suggested that pupils make graphs of relations. As the pupils observe many different kinds of graphs, those graphs characterizing functions will stand out in sharper focus.

Also important in mathematics are the special relations called order relations, such as *more than* and *less than*. These are used when such concepts as *heavier than*, *longer than*, *darker than* or *thinner than* are being considered. Measurement such as that of time, capacity and length consists of ordering the quality to be measured and then assigning numbers to correspond to that ordering. Thus the numerical order relation makes precise the intuitive one.

The activities in this strand include suggestions for introducing pupils to relations in general and to the special relations discussed above. As with other strands, teachers will need to select those which are appropriate for their class and supplement them as necessary. It should be reemphasized, however, that familiarity with relations in general should precede formal work with special relations.

Mathematics can be viewed as an entity of systems, each consisting of the following components—sets of elements, or basic units such as whole numbers, rational numbers, or points; relations or comparisons of these elements, such as *equal to, greater than* or *congruent to;* and operations such as multiplication or set union. Therefore, recognizing and using relations constitutes a basic activity in which the pupil must engage in order to understand the concepts included in that strand.

Relations and Functions

Objectives

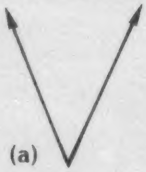
The student should be able to do the following.

1. Classify elements of a set according to specified properties
2. Demonstrate correspondences (a) one-to-one, (b) one-to-many, (c) many-to-one and (d) many-to-many
3. Apply equivalence relations to elements such as to fractions, ratios, percents and measures of geometric figures, e.g., set up a proportion
4. Find some pairs of elements when a relation is given, e.g., given the relation *square the number* some pairs of elements are (2, 4), (3, 9), (5, 25), (10, 100)
5. Find the missing element of a pair when one member of the pair and the relation are given, e.g., given 4 and the relation *multiply by 3*, then the missing element is 12
6. Find the relation when a set of ordered pairs is given, e.g., given the set of pairs (3, 6), (4, 7), (5, 8), (7, 10), the relation is *add 3*
7. Order a set of elements according to a specified relation, e.g., *greater than, less than*
8. Use the addition and multiplication properties of equality to solve one-variable open sentences, e.g., if $A = l \bullet w$, then $l = A/w$ or solve for x in the proportion $8/6 = 20/x$
9. Use a graph on a number line to represent
   (a) a number or numbers
   (b) the solution set of an equation
   (c) the solution set of an inequality
10. Use a graph in a coordinate plane to represent
    (a) ordered pairs
    (b) the solution set of an equation
    (c) the solution set of an inequality
11. Interpret a graph (a) on a number line and (b) in a coordinate plane
12. Determine if a given relation is a function
13. Represent a function using a
    (a) sentence
    (b) formula
    (c) mapping
    (d) table
    (e) graph

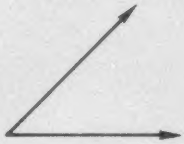
Geometry

Introduction

Geometry in the early grades is often characterized as a study of the names and simple properties of common geometric figures such as triangles, rectangles and circles. Although these concepts are certainly a part of geometry for the middle grades, a more reasonable characterization would include the study of relations among sets of points. Consider the difficulty many pupils have in learning to use a protractor. Given the task of measuring the angles in the accompanying illustration pupils may report 120° for (a) and 60° for (b).



(a)



(b)

They are not likely to reject these answers unless they recognize that the angles are congruent. That is, students can make use of a relationship if they first of all recognize it. The activities in this guide are provided to give opportunities for pupils to discover relationships and to recognize the conditions which give rise to those relationships.

For this reason it is helpful if the study of geometry below the high school level is thought of as exploration of space. Exploration means searching, probing or making discoveries. The activities suggested are intended to give a pupil the sort of firsthand experience which can properly be called exploration. By pulling, turning, sliding or folding students learn to predict the appearance of geometric figures under different conditions; by designing a pattern for a model students learn for themselves what parts must be assembled and what arrangements of these parts will and will not work and by creating larger and smaller copies they discover properties which are independent of size.

Explorers do not set out with a ready-made itinerary and an immutable timetable, but it is helpful if they can consult from time to time with someone who knows the terrain and can give them some advice. In exploring geometry, the teacher can be this consultant by suggesting other things to try, posing the right sort of question and encouraging the pupil to find answers by experimenting with objects or representations of objects.

Finally, explorers take notes as they go but write the final report at the end of their journey. Thus, in geometry, formal definitions and precise statements of generalizations should be the culmination rather than the beginning of a journey through a topic. The teacher plays a vital role here, for the young explorers cannot discover names for what they have seen; these must be supplied by the teacher. The pupil pulls, turns, slides and folds pieces of balloons or pieces of paper with drawings on them and sees how drawings appear afterward. In discussing what they have found, they will find it helpful to have a single word to use instead of repeating the sequence, "Pulling, turning, sliding or folding." The teacher can then supply the word **transformation** or **motion.** Students, however, do not need to know this word before they start out; they can learn what the pictures look like without ever having heard the word. Likewise, in discussion it may be helpful to have a word to use instead of "all these ways the drawing might look if I turned the paper," and again the teacher can supply the word **equivalent** for the set of drawings in question. The role of language in geometry should be to facilitate learning and communication; word study should not be an end in itself. A note of caution, however, is in order. Some words in geometry are also used in everyday life, but with a slightly different meaning. In these cases, it may be wise to call the pupil's attention to the familiar uses of the words and point out the restrictions relative to geometry. A good example is the word straight. "Going straight down the road" may not be straight in the geometric sense.

Since this volume of the guide is intended to span grades five through eight, some of the activities outlined here are too difficult for fourth graders. Teachers will need to select and pursue those activities they choose to a depth appropriate for their class. Provision has been made in some activities for considerable depth of development. Supplementary suggestions can be found in the references given in the media listing.

Geometry

Objectives

The students should be able to do the following.

1. Identify and name a point, line, line segment, ray and angle
2. Identify and classify closed curves in a plane such as a square, rectangle, other parallelograms, triangle and circle
3. Identify and classify three-dimensional objects such as prism, pyramid, cone and cylinder
4. Identify shapes that are alike under stretching, shrinking or bending
5. Identify shapes that are alike under rotations (turns), reflections (flips) or translations (slides)
6. Determine relations between point sets or between geometric figures such as inside, outside, parallel, perpendicular, similar, congruent
7. Use relations between and among point sets or geometric figures to deduce other relations, e.g., congruence of alternate interior angles to prove that two lines are parallel
8. Determine lines of symmetry
9. Measure angles and classify as acute, right or obtuse
10. Use formulas to solve geometric problems involving perimeter, area and volume
11. Identify a center, radius, diameter and other chords of a circle
12. Apply the formulas for finding (a) the circumference of a circle and (b) the area of a circle
13. Solve simple geometric problems by using properties of right triangles
14. Solve simple geometric problems by using properties of similar figures, e.g., indirect measurement
15. Solve simple problems which require the basic ruler and compass constructions
16. Associate three-dimensional objects with their two-dimensional representations

Measurement

Introduction

Measurement is the process of relating a number to a property of an object or a set. Measure is a number which tells how many specified units have been assigned or determined. Measurement activities should include activities related to time, weight, capacity and volume, area, and length.

Counting, the measure of *how many*, yields the only exact measure. In this strand, counting is used to find the number of units of measure. Activities of this type should precede those in which the pupils find measurements from reading scales of measuring devices. In using a device, the unit of measure, which has the same property of the object to be measured, is compared to the object. The resulting measure is never the same as the true measure even with the most accurate instrument.

Measurement is a process of doing. It should be studied in this manner at all grade levels and should develop through a number of stages (1) making gross comparisons of objects or sets, (2) using units of measure devised by the students and (3) using units of measure called standard units. After these activities, the students should have opportunities to derive geometric formulas and to use them successfully. Even though students in the middle grades have had some experiences in stage one in the primary grades, some students may need additional activities in the first stage in order for them to realize that measurement is a comparison.

The second stage should develop from the first stage. Units of measure will be needed to relate comparisons. The first ones should be improvised or homemade ones. Students should be encouraged to measure many objects with their homemade units. The third stage should be introduced only after the students have had experiences in the earlier stages and have seen a need for standard measure. The teacher can devise situations where the students will see that a standard measure would be more practical than individual students using their own.

An illustration of the various stages in teaching the concept of area might be one of these.

1. Have the students use a small circle to measure the area of an object like the top of their mathematics books. They will quickly see that a more appropriate measure would be a square or a rectangle.
2. They could then place the squares on the object to measure it. If a drawing is used, they could glue the squares on or draw them on the object which is much easier than just holding them on as they count. They could also refer to them later in deriving the formulas. It would be helpful if they use different size squares or rectangles to measure so that the concept of standardness could be discussed later.
3. An acetate grid of centimeters or inches could then be placed over the object as an "easier way" to count. This could be done after the class has decided that they need to use a standard measure. They could continue to draw if they preferred.
4. They could discover the formula as the number of units in the length times the number of units in the width and could see several examples of this in their previous work.

The students should have experience in measuring and performing computations in both the English and metric systems. The emphasis should be placed on working within each system and so much on the conversion from one system to another. However, approximate relationships of the most often used units in the two systems should be discussed.

They will need more opportunity for actual measuring experiences with metric units since these are not as familiar to them.

The types of activities for measurement given in this strand are not all inclusive. Pupils in the middle grades should be introduced to precision of measurement and to other measurement topics, such as the following.

- Speed, measured in miles per hour or feet per second
- Light—brightness measured in footcandles or magnitudes of stars
- Heat—the intensity using Fahrenheit and Celsius scales and the amount of heat using the units BTU and calorie
- Sound—the intensity and pitch
- Pressure of both air and water
- Electricity—pressure measured in volts, amount of energy measured in kilowatt-hours and rate of flow measured in amperes
- Hardness of rock or other substances measured by a scale of hardness of minerals

Measurements

Objectives

The student should be able to do the following.

1. Determine a time interval between two events
2. Determine (a) final time reading or date if given the initial reading or date and the time interval and (b) the initial time reading or date if given the time interval and the final reading or date
3. Determine the mass (weight) of an object using (a) nonstandard units and (b) standard units
4. Make a reasonable estimate of mass (weight) of an object or substance and verify the estimate
5. Determine capacity or volume by counting (a) nonstandard units and (b) standard units
6. Use experimentation to derive the formula for the volume of a rectangular container
7. Make a reasonable estimate of capacity or volume of a rectangular container and other three-dimensional objects and verify the estimate
8. Derive formulas for the volume of three-dimensional objects other than rectangular ones by experimentation and by the application of formulas
9. Determine the area of a rectangular region by counting (a) nonstandard units and (b) standard units
10. Derive the formula for the area of a rectangular region by experimentation
11. Apply the formula for the area of a rectangular region to derive formulas for the area of other regions
12. Make a reasonable estimate of the area of a region and verify the estimate
13. Determine a length or distance using (a) nonstandard units and (b) standard units
14. Make a reasonable estimate of a length or distance and verify the estimate
15. Derive formulas are finding perimeters simple closed curves
16. Select the appropriate type of measurement needed for a given problem situation, e.g., length, area, volume
17. Select the appropriate formula(s) for finding the measurement for a given situation, e.g., area, perimeter, circumference, volume
18. Identify the precision appropriate for a specified measurement
19. Apply measurement to other fields such as science and social science

Probability and Statistics

Introduction

Basic to statistics are the techniques of collecting, organizing, summarizing and analyzing data. Once the data are summarized and analyzed, the predictions that are made become the study of probability. Therefore, statistics and probability are studied together.

Because statistics is used in areas of science and social science such as insurance, marketing, astronomy and genetics, it is necessary to acquaint pupils early with the use of statistics. In addition, one of the easiest ways to develop problem solving techniques is through the collecting of statistical data that are real to the pupil. Probability and statistics are good vehicles by which the teacher causes the pupil to make choices, inferences and valid judgments.

The student in the middle grades, as well as the primary grades, should have many experiences collecting, recording and exhibiting data as these topics form the basis for the study of statistics. The pupil should then begin to have some experiences in interpreting data through the study of distributions, range, central tendency and deviation.

Making predictions from the data collected necessitates the study of probability. Before formalizing the mathematical definition of probability, the pupil should engage in many game-like experiments in which he studies the chances of related outcomes. In identifying outcomes and assigning probabilities, pupils learn the concept of sample space and some counting shortcuts. These counting techniques are particularly helpful with experiments involving a great number of events.

Probability and Statistics

Objectives

The student should be able to do the following.

1. Identify different techniques for collecting data
2. Sort out what are relevant and what are irrevelant data
3. Demonstrate the need to collect a large enough sample of data to be representative
4. Distinguish between a biased and an unbiased sample of data
5. Give an example of how sampling affects the interpretation of the data
6. Identify various methods that may be used to record raw data and be able to select an appropriate way to represent a particular set of data
7. Construct and interpret graphs
8. Make predictions based on data represented in a chart or graph
9. Find the range and measures of central tendency, i.e., mean, mode and median, of a set of data
10. Identify uses and misuses of measures of central tendency
11. Compare a given number to the mean of a set of numbers by finding its deviation from the mean
12. Identify the sample space for a simple experiment by
    a. describing or tabulating outcomes
    b. tree diagrams
    c. counting principles
13. Find the probabilities of outcomes including those that are
    a. certain to occur
    b. certain not to occur
    c. equally likely
    d. not equally likely
14. Use probability to decide whether or not two outcomes are equally likely
15. Use probabilities to make predictions
16. Find the relative frequency of an outcome in an experiment
17. Compare predictions based on probability with actual results of an experiment

Processes

Why should the teacher emphasize processes? What is meant by processing in mathematics education? What implications for teaching can be found in a process-oriented emphasis?

What Is Processing

Consider your own knowledge of mathematics. You know a fairly extensive set of ideas, including concepts (e.g., fraction, length, rectangle, prime number or multiplication) generalizations (e.g., $A = r^2$, laws of exponents, or the distributive property), procedures (e.g., how to find the greatest common factor of two counting numbers, how to construct parallel lines or how to find the quotient of two fractions), and solutions of problems (e.g., how many diagonals in a 20-sided polygon?). These aspects of your mathematical knowledge may be easily recognized.

There is, however, another dimension to your knowledge which is often more elusive. As you developed your concepts, generalizations, etc., you were also developing your knowledge of *how to construct* new ideas in mathematics for yourself. These involved many processes, perhaps including abstracting, generalizing, simplifying, patterning, differentiating, guessing, conjecturing, assuming, recalling, concluding and questioning. Without the activation and refinement of these mental operations your growth in mathematics would surely have been hampered and limited. It is through the *use* of such processes in building knowledge that the processes become more functional, more powerful and more readily available for later use.

It may be possible to recognize and, to a certain extent, control the use of certain cognitive processes. As humans, we are able to reflect upon the ways and means by which we come to know. Thus, by thinking back over the steps followed, the self-questioning used, the decisions made, the reasoning employed and the results obtained, we may come to understand more clearly what mental tactics we may have followed. Through such reviews of our experiences we may be able to understand the essential processes. What was used at this time to build up a solution or understand a concept may be strategically helpful in a future task.

How to Emphasize Processing

How might middle school mathematics teachers promote the child's awareness and application of such processes? Perhaps a key to such teaching lies in the act of looking back upon a learning episode with the learner. Some advice for doing this will appear throughout this guide. But intuitively it may seem sensible to conduct this looking-back phase as close in time to the actual events about which you are reflecting. Thus, many teachers may point out processes throughout a lesson, or summarize and discuss the processes used near the end of a lesson. Of course, sometimes you might look back over an entire series of lessons in order to emphasize certain tactics used in processing the tasks of the mathematics learned during this period.

Labeling the process being used may not be so important as the effort to help the child become more aware of what or how he/she may be thinking at a particular step. Furthermore, the extent to which you treat processes will certainly vary with each child. One caution may be to avoid becoming too centered upon identifying processes used. You will need to develop a balance in your focus upon thinking about such cognitive processing. As you become more attentive to thinking processes, you may find that many times you will make note of how you or a child may be approaching a task, yet you may choose not to say anything about it. In our struggle to individualize our instruction it may be important to make process-oriented observations about a child in order to understand the child's thinking as a basis for the instructional decisions we might make about that child.

Why Promote Processing?

In the foregoing discussion we have advocated the identification and articulation of imporant cognitive processing to be found in the child's or the teacher's thinking as they do mathematics. Let us express one perspective about mathematics learning and teaching which may provide a foundation for choosing to emphasize processes. What is mathematics? What is knowledge? These two fundamental questions can be approached as follows. Mathematics, as a discipline, can be viewed as an extensive, ever-growing body of information—something **out there** to be studied, learned, ignored, hated, etc. But it can also be described as a way of thinking about, or processing, problematic situations. In these terms mathematics is a human construction, built-up by individual thinkers.

One consequence of such a view is that each person must be an active, responsible builder of his/her own knowledge. It is impossible for any well-intending person, such as a teacher or parent, to **give** knowledge to the child. Through mathematical instruction, stimulation and guidance are provided the learner as he/she builds and rebuilds ideas and develops new ideas. So the focus should be upon showing the students their responsibilities for constructing their own knowledge. Of course, the role of the teacher should be to assist in the child's constructions by providing tasks, advice, and encouragement. But to approach teaching mathematics as a task of simply displaying information, such as a page in a textbook might be designed to do, may not be sufficient. If a teacher always demonstrates procedures or solutions without allowing students to construct their own responses, important experiences in building knowledge may be lost for the child. Tasks must be set for learners in such a way that individuals have opportunities to construct **their** own processing responses.

Implications for Teaching

Of course, throughout the child's experiences the teacher will often serve as the principle source of information. But we must be selective in how we give information when what may be needed is guidance in using what the child already knows. This may become clearer if we recognize that what we are trying to teach includes not only the observable material (e.g., definitions, written steps, figures, problem statements) but also the means (i.e., the processes) for dealing with this material. We must be aware that students are learning about how-to-learn (e.g., how to solve, how to generalize, how to apply, how to feel toward mathematics) when they are learning a particular bit of content.

Thus, a teacher needs to recognize that, ultimately, each individual must construct the knowledge which he/she comes to possess. We can help by providing a proper classroom atmosphere for such initiatives from the child. There must be a challenging but open and accepting climate. Risks can be taken and errors can be made since students benefit from trial and mistakes in the teacher's efforts to construct his/her own solutions, generalizations and procedures. At the same time, the tasks must be motivating and the attitude of productivity must be valued, for we seek to have students enjoy and respect their own efforts and achievements. If the teacher approaches each new mathematical task as being a sensible response to a reasonable situation, students may feel challenged yet confident that they will build useable knowledge.

Tasks of the Middle School Mathematics Teacher

We have noted that learning is a person-centered, constructive building process. In order for learning to occur, students must be *active builders* of their own conceptions. Instruction is a context as well as a process for stimulating and guiding these builders in terms of their own constructive processes.

Effective instruction of mathematics can be viewed in terms of the many varied tasks which engage the teacher. What are the broad instructional tasks which confront middle school mathematics teachers? Basically, teachers must help students to construct complex constellations of *ideas* and *feelings* about mathematics. In addition students must be helped to learn *methods* and *strategies* for constructing new mathematical ideas. Some aspects of these two overall tasks are represented below.

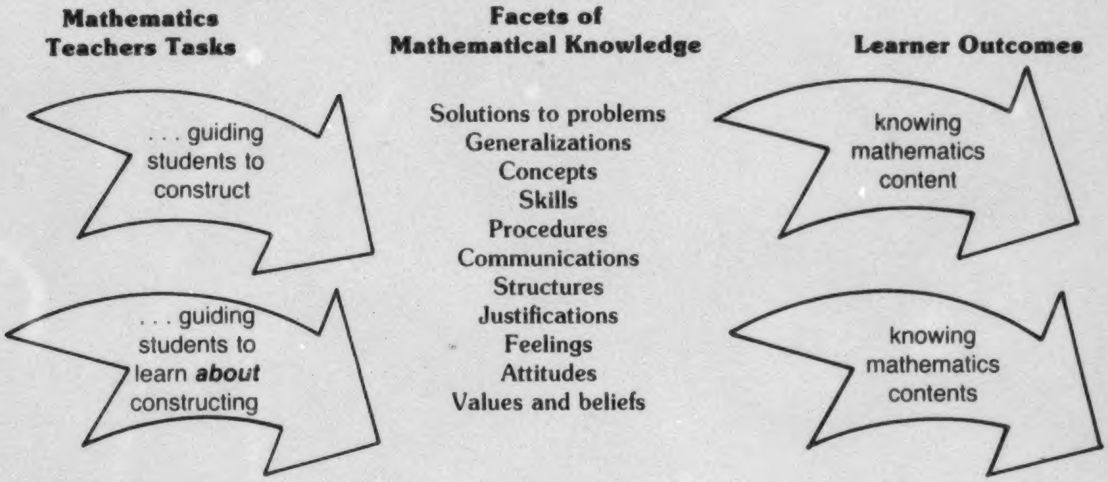

Mathematics Teachers Tasks
Facets of Mathematical Knowledge
Learner Outcomes
. . . guiding students to construct
. . . guiding students to learn about constructing
Solutions to problems
Generalizations
Concepts
Skills
Procedures
Communications
Structures
Justifications
Feelings
Attitudes
Values and beliefs
knowing mathematics content
knowing mathematics contents

Figure S-1

Mathematics teachers must view their own instructional tasks so that the child's learning includes **particular** items of knowledge (such as a concept of area or a procedure for adding fractions) as well as **generalizable** processes (such as an ability to generalize, or to evaluate the reasonableness of an answer). It is an important purpose of this curriculum guide to encourage and assist teachers in developing instruction which includes both emphases.

Traditionally, mathematics teachers have focused their instruction on the learning of mathematics content. In many mathematics classrooms students do learn about the processes for building mathematical knowledge, because the children obviously use the processes as they are constructing a new idea. Yet the learners may be unaware of what helps them to know a new algorithm or concept because such process-oriented considerations are only implicit. Other teachers give explicit, conscious attention to the strategies of thinking which could be used in building the idea. These teachers often focus discussion on the mental operations by helping students to "look back," reflecting upon what worked well today which might be useful with new ideas to be constructed tomorrow.

In the following sections we will identify and illustrate specific teaching methods which incorporate concerns for both aspects of teacher tasks: guiding students to constructions and to learning about constructing.

Teachers use processes, too

Throughout this guide we have encouraged a constructive approach to learning and teaching of middle school mathematics. Additionally, we have emphasized the importance of process-oriented strategies and outcomes of learning. There are equally **generalizable** teaching processes which teachers may use to complete the instructional tasks discussed in the previous section.

What are some teaching processes for helping students with learning and learning-how-to-learn mathematics? Any attempt to identify comprehensively the processing which an effective teacher employs helps one to appreciate how truly complex teaching is. Some important aspects are shown below.

Teaching Processes

"Guiding" Cluster

- Challenging
- Questioning/Answering
- Experimenting/Searching
- Observing
- Hypothesizing/Conjecturing
- Justifying/Proving
- Assuming
- Guessing/Intuiting
- Simplifying/Interpreting
- Generalizing/Concluding

"Telling" Cluster

- Stating/Asserting
- Explaining/Clarifying
- Describing
- Exemplifying/Instancing
- Characterizing
- Demonstrating/Modeling
- Defining

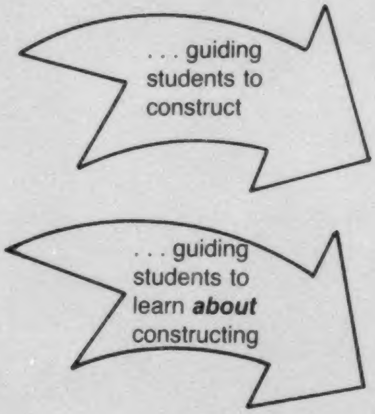
"Managing" Cluster

- Planning
- Directing
- Prescribing/Assigning
- Organizing
- Summarizing/Reviewing
- Evaluating/Reacting
- Adjusting/Adapting

"Feeling" Cluster

- Motivating
- Valuing/Recognizing
- Praising/Rewarding
- Encouraging
- Allowing/Risk-taking
- Judging
- Imposing/Expecting

Teacher Tasks


. . . guiding students to construct
. . . guiding students to learn ***about*** constructing

Though this list is quite comprehensive, there may be additional teaching processes which you can find in your own instruction. We have organized the identified teaching processes into four clusters for discussion purposes, but your interpretations might easily result in different groupings. In some cases complementary or related processes are noted together, such as **experimenting/searching.**

This list of possible teaching processes is not intended to prescribe what you are to do. It is not a checklist. Rather, it is intended to help you reflect upon, and describe, your own teaching emphases.

Think about your own instruction as you work with children. Consider these questions.

- Which teaching processes do you engage? Are there any which you do not employ?

  You may find that your teaching generally involves only a few of the processes. For example, your lessons may include stating, demonstrating, clarifying, answering, assigning, recognizing and evaluating. You may find that you never use challenging, experimenting, justifying, characterizing or summarizing.

- Which processes are dominant in your teaching?

  Some teachers find that their lessons involve mainly questioning/answering based upon textbook developments. Others, primarily use stating and modeling/demonstrating at the chalkboard. Still another dominant pattern may employ only assigning, describing and evaluating.

- Do you predominately draw from a particular cluster of teaching processes?

  You may realize that most of your lessons center upon the **telling cluster** as you make assertions or give definitions, describe, give examples, point out characteristics and demonstrate or model procedures. On the other hand, perhaps your teaching activity is focused upon giving directions, assigning tasks, reviewing completed assignments and evaluating progress, all aspects of the **managing cluster.**

- In what ways are different teaching processes used to individualize your instruction? Are particular processes always (or never) used with particular children?

  Teachers often behave differently with different students or classes. You may find that with students thought to be less able you never engage in experimenting, conjecturing or generalizing, in spite of the fact that they may need these experiences more then other students.

- How do you use different **teaching** processes as a student passes through various phases of construction of an idea?

  Perhaps one of the most sensitive ways we can provide for individuals is to recognize that each learner may go through differing phases as he/she builds a new item of knowledge. Some teachers may place early emphasis upon challenging, questioning/answering and conjecturing when a child is confronting an unfamiliar situation. Others may resort to mainly explaining, modeling and prescribing in these early stages.

- How do you use these teaching processes to allow and encourage constructive approaches to learning?

  With a belief in constructivism in education this becomes a crucial concern. It would seem that the following considerations may be important.

  - Students need to realize their own responsibility and control over the building up of their knowledge. You can help by motivating, allowing risk-taking and expecting as the climate for learning.
  - You may need to encourage students to formulate and share their own interpretations or procedures. Of course, these may be incorrect, whereupon you may guide them in modifying or rebuilding their idea.
  - No matter how an initial formulation of an idea is stimulated, it will be crucial to attempt to "get in touch" often with how the student is thinking about the idea. This will probably require personalized questioning and observing in order to get the student to reveal how he/she comprehends the idea.
  - Throughout your own awareness of instructional processing is the associated goal of guiding students to a deeper awareness and understanding of their processes used in constructing their own knowledge. You would probably help students to mature in their explicit knowledge of processes by modeling your own thinking, using inquiry processes in the guiding cluster and reflecting back with

students upon which processes have been used in building up a particular solution, generalization, procedure or concept.

- What relationships do you see between these teaching processes and the learner processes?

  Often the processes used by the teacher are directly related to the processes adopted by the learner. If students can participate in the teacher's questioning, simplifying or exemplifying, they may engage such strategies in approaching other learning tasks. A teacher who can approach a concept or procedure with an open, questioning attitude, allowing variations in the interpretations or constructions, will encourage a flexible style. On the other hand a teacher who instructs primarily by autocratic telling, directing and prescribing may instill an attitude that mathematics is not something to be acted upon but is information to be received from an authority. Be aware that sometimes students may adopt processing strategies quite unlike those modeled or advocated by the teacher. These learner-initiated processes should be considered as important sources of information in your efforts to know your students. Ask your students to explain how they think about a particular problem or procedure. Whereas a student's way of thinking may not be mathematically sophisticated, it is often intuitively sound and worth sharing. Knowing how a student processes a task provides a basis for guiding the refining or correcting which may be needed.

Review again the Facets of Knowledge List in Figure S-1. Do you employ different teaching processes for helping a child build a particular aspect of knowledge? You may be aware that certain mathematical tasks lend themselves to a particular development or approach. For example, you may always approach the teaching of generalizations by asserting a statement of the generalization, clarifying terms, rephrasing it into common language, developing instances, justifying its truth and practicing its use. Creative teachers may be able to recognize that any item of knowledge may be constructed by a variety of process options. For example, you may try to help students to design their own algorithms for a mathematical task, from which you then guide them toward modifying and improving their invented procedure.

Indeed, it can be a complex and strenuous matter to examine the details of one's own teaching. Such relfection must be deliberate, for it appears to be more common to teach without conscious attention to the choice and conduct of such teaching processes. Becoming more aware of what you do may hold the promise of becoming a more effective teacher.

Process skills as discussed in this guide includes the thinking skills listed and discussed in a section of the *Essential Skills for Georgia Schools*, a publication of the Georgia Department of Education. A copy of the taxonomy and definitions stated in the document is included here for your use.

In the *Essential Skills for Georgia Schools* a listing of concepts/skills is given for each of the major process skills. The grade level intervals, K-4, 5-8 or 9-12, at which each concept/skill might be introduced, developed or reinforced is indicated in the listing.

Thinking Skills Taxonomy and Definitions

In selecting the taxonomy of thinking skills for Georgia public schools, the developers of this manual have drawn upon the current knowledge in the field as represented by the foregoing discussion. Definitions of the chosen terms follow.

- **Recalling**—recognizing information previously encountered such as facts, concepts or specific elements in a subject area.

  Identifying—*ascertaining the origin, the nature or definitive characteristics of an item.*
  Observing—*obtaining information by noting, perceiving, noticing and describing. Observation may involve looking, listening, touching, feeling, smelling or tasting.*
  Perceiving—*directly becoming aware of an item through any one of the senses, especially to see or hear.*

- **Comprehending**—understanding information that has been communicated.

  Translating—*changing information from one form to another. This is judged on the basis of accuracy, that is, the extent to which the original communication is preserved although the form has changed.*
  Making analogies—*inferring that if two things are known to be alike in some respects then they may be alike in other respects.*

- **Hypothesizing**—assuming, making a tentative explanation.

  Predicting—*telling or declaring beforehand.*
  Imagining—*forming a mental image, representing or picturing something to oneself.*

- **Applying**—putting to use.

  Clarifying—*making something easier to understand.*
  Testing hypotheses—*trying out ideas for possible solutions.*
  Defining operationally—*ordering ideas into a step-by-step plan.*
  Decision making—*choosing the best or most desirable alternative.*
  Projecting consequences—*defining further steps toward probable solutions or identifying cause/effect relationships.*

- **Analyzing**—systematically or sequentially breaking down a concept, problem, pattern or whole into its component parts so that the relationships between parts are expressed explicity.

  Comparing—*determining similarities and differences on the basis of some criteria.*
  Classifying—*placing elements (e.g., objects, ideas, events) into arbitrarily established systems of groupings and subgroupings on the basis of common characteristics.*
  Selecting—*choosing an element from a set of elements (e.g., objects, ideas, events) on the basis of some criteria.*
  Associating—*relating elements (e.g., objects, ideas, events) either as given or as they come to mind.*
  Inferring—*drawing a conclusion based on facts or evidence.*
  Interpreting—*expressing meaning of or reaction to an experience, e.g., a reading, an excursion, a graphic representation, a work of art.*
  Qualifying—*describing by enumerating characteristics.*

- **Synthesizing**—putting together elements by arranging and combining them to form a structure, pattern, or product.

  Summarizing—*expressing a brief or concise restatement.*
  Generalizing—*formulating or deriving from specifics (to make universally applicable) a class, form or statement.*
  Formulating concepts—*originating or expressing ideas.*
  Intergrating—*forming into a whole, uniting.*

- **Evaluating**—making judgments regarding quantity and/or quality on the basis of a set of criteria.

  Justifying—*showing adequate reason(s) for something done.*
  Imposing standards—*assuring equal comparison with established criteria.*
  Judging—*forming an idea or opinion about any matter.*
  Viewing internal consistency—*understanding that all the parts of a process fit together.*
  Valuing—*establishing worth or esteem.*

Planning Instruction

How to Begin
Personalizing Instruction
Special Considerations
Using Support Systems
Sample Activities

Planning Instruction

Once goals and objectives based on a given philosophy have been determined, the planning for instruction can begin. There are obviously many plans that need to be developed at every level—the state, the school system, the school building, the classroom. Suggestions have already been provided for the school system and individual school in the part of the guide entitled "Developing a Middle Grade Mathematics Curriculum." The suggestions in this part of the guide are written primarily to assist classroom teachers as they plan the daily instruction for students in the middle grades. The first section gives suggestions on how to begin to plan daily instruction for the middle grades learner. The second section deals with the most basic of all concerns in planning—providing instruction that attempts to meet the needs of each person who is a member of that individual classroom.

Following this section are several articles that present some very specific considerations. The impact of technology on teaching techniques and the curricular needs of students is the thrust of the first article. The next article discusses difficulties that students encounter in reading mathematics and in solving word problems. Many students and teachers in the middle grades have anxiety about mathematics which prevents optimum learning from taking place. The third article addresses this concern. The next article deals with students who have special difficulties that affect their progress in mathematics. There are discussions of areas of exceptionality and recommendations are made for modifying techniques to meet the individual needs of these students. The last article on special considerations provides suggestions for planning instruction for students who are gifted in mathematics.

The task of providing effective instruction for all of the individuals in a classroom in the middle grades can become an overwhelming one for teachers unless they have an adequate support system. The intent of the section on support systems is not only to identify many of the sources of support but also to make suggestions to teachers for more effective use of the various aspects of available support.

The section on strategies includes a discussion of tasks that face teachers in their daily instruction. Several classroom episodes are included to illustrate these tasks as they relate to a given classroom situation.

Developing appropriate activities to meet all of the needs discussed in the other sections of this part of the guide entitled "Planning for Instruction" is a time consuming idea. The remainder of this part of the guide contains suggestions for such activities as well as some activities that have already been developed. There are a variety of formats for the activities since some teachers might prefer many ideas from which they could select those they wish to develop while other teachers might find the fully developed activities more useful either as they appear in the guide or as a sample for developing additional activities. In addition, since teachers also have individual needs and teaching styles, some formats of the activities may be useful during one time period, while others may be preferred at other times. If the task card activities are used as they appear in the guide, there should be some coordination among staff members as to which activities will be used at the various grade levels or modifications of the cards could be made and agreed upon for different grade levels, e.g., deletions or extensions. Clearly this agreement is necessary so that students will not be given identical activities for several consecutive years.

Planning for instruction is an important task for every classroom teacher. Understanding the complexity of the individuals in the middle grades, their special needs and how to begin to meet those needs will make the task more manageable.

How to Begin

Planning for instruction in a broad sense includes setting goals, selecting materials, using support systems, assessing and other activities discussed in preceding sections of the guide. This special section of the guide has been developed to give added focus to the concerns of teachers as they attempt to plan daily instruction for the students assigned to their classrooms. Many groups of teachers and others may also be involved in setting program goals in a local school system and in selecting the basic mathematics textbook series. Class size, schedules, assignments of individual students, etc., are often determined by administrators. All of these decisions are extremely important in the provision of quality mathematics instruction for students. None of these, however, exceeds in importance the actual planning for instruction that is carried out by the individual classroom teacher.

The expansion of subject area content needed for a technological society, the call for increased attention to problem solving and other higher level skills, and the unbelievable number of tasks asked of teachers that take away instructional time, are factors which contribute to a national concern for maximizing the time spent in mathematics instruction for each student.

The National Council of Teachers of Mathematics appointed a study committee to provide direction for the teaching of mathematics in the 80s. The results of the study were published in *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (National Council of Teachers of Mathematics, 1980). Included were eight major recommendations. The fourth recommendation concerns standards of both effectiveness and efficiency which must be applied to the teaching of mathematics. In stressing the significance of what happens during the hour of mathematics instruction the report stated that

> Instructional time is a precious commodity. It must be spent wisely. Learning is a product of both the **time engaged** in a learning task and the **quality** of that engagement. Teachers must employ the most effective and efficient techniques at their command. They must apportion instructional time according to the importance of the topic, recognizing that the value of a skill or knowledge is subject to change over time.
>
> Modern technology and educational theory and research have made accessible to today's teacher approaches, materials, and strategies that were not previously available. Teachers at all levels must learn to use this enriched variety of instructional techniques, materials, and resources to teach mathematics more effectively. (NCTM, p. 11)

The Georgia Board of Education and the Georgia Department of Education have demonstrated their belief in the significance of teacher-controlled activities by the development and implementation of Performance-based Certification for Teachers. These agencies strongly support the concept of the significance of the teacher-controlled activity within the classroom.

With the assistance of many professionals from various educational settings throughout the state, the Georgia Department of Education and the State Board of Education have identified specific generic competencies which are essential for effective teaching in the Georgia classrooms.

1. Plans instruction to achieve selected objectives.
2. Organizes instruction to take into account individual differences among learners.
3. Obtains and uses information about the needs and progress of individual learners.

4. Refers learners with special problems to specialists.
5. Obtains and uses information about the effectiveness of instruction to revise it when necessary.
6. Uses instructional techniques, methods and media related to the objective.
7. Communicates with learners.
8. Demonstrates a repertoire of teaching methods.
9. Reinforces and encourages learner involvement in instruction.
10. Demonstrates an understanding of the school subject being taught.
11. Organizes time, space, materials and equipment for instruction.
12. Demonstrates enthusiasm for teaching and learning and subject being taught.
13. Helps learners develop positive self concepts.
14. Manages classroom interactions.

Specific suggestions have been included in the guide which are intended to provide assistance in each of these competency areas to teachers of mathematics in the middle grades in the state. For example, these.

1. The evaluation section should provide assistance with competencies 3 and 5.
2. The section on support systems addresses competency 4.
3. The section on the nature of the middle grades learner should assist the teachers in competencies 2, 7, 9, 13 and 14.
4. The section on personalizing deals very specifically with competencies 2, 3, 8, 9 and 11.
5. Suggestions for competencies 7, 9, 12, 13 and 14 are included in the five articles under the section "Special Considerations."
6. The section of the guide which contains very specific help to the teacher in planning learner tasks and teacher tasks should provide assistance to the teacher on almost all of the competencies. Competencies 1, 2, 6, 8, 9 and 11 are given special attention in this section.

Although all of the sections of the guide provide assistance to the teacher on several of the competencies, there are three sections that deal almost totally with the planning that is done on a classroom by classroom basis. The section on structures for individualizing is devoted entirely to topics such as grouping of students, use of learning centers, learning activity packages, mathematics labs and the like. Another section provides a multitude of ideas for given topics normally taught during the middle years (See the section on "Tasks for Learning and Teaching Middle Grades Math.") The section on support systems should provide assistance to the teacher in the area of planning for instruction.

In summary, the planning of daily instruction by **individual classroom teachers** will probably influence the progress made by students **more significantly than any other single factor.** This entire guide has been developed to assist classroom teachers as they plan—with the goal of providing optimal conditions for meeting the criteria of effectiveness and efficiency in mathematics instruction in the middle grades.

Personalizing Instruction

There are several factors that complicate the process of dealing with individual differences in learning during the middle grade years. Although students already possess different abilities, attitudes and learning styles when they enter school, these differences are magnified during the middle grades. In addition to this expanded diversity, many students during the middle grade years may temporarily lose their enthusiasm for learning. The academic growth of these students therefore is often dependent on external motivation or a revitalization of internal motivation. Unless the instructional program is designed to meet individual needs, there is little expectation that either will occur. If, on the other hand, the day to day instruction is planned and executed with high regard for each individual in the classroom, academic growth should be expected.

Personalized instruction in this section is defined in a very broad sense to indicate instruction that is designed to meet the needs of individual students in the classroom. One of the greatest challenges in teaching at any grade level may well be the challenge of attempting to personalize or individualize instruction to meet these individual needs.

In recent years, the term **individualized instruction** has taken on a very special meaning. It has been used by some educators to describe a program where students use a sequence of learning packets or units and move through the material at their own pace. Passing a pre-test and/or post-test is a prerequisite to progressing to the next unit. The curriculum itself is structured and sequenced and is the same for each student even though each student moves through the program in a continuous progress mode. By the late 1960s the merging of dual school systems, social promotion and the tendency of governmental agencies to require a return to heterogeneous classrooms magnified the diversity of academic levels in a given classroom and promoted this concept of a separate program for each individual student. As federal programs were funded for curriculum development and implementation of programs, many systems for self-paced instruction were developed and marketed to schools throughout the nation. Textbooks were revised so that they could be used in this manner. As schools purchased these programs, many classrooms were transformed into reliance on one particular mode of instruction. If any flexibility remained for teachers to use other methods of individualizing, these teachers often changed to this format for a variety of reasons. Although many of these programs violated one of the basic philosophies of most educators, i.e., that no one method is best for all children and all teachers, they continued to gain acceptance as **the** answer to the challenge of meeting the individual needs of students in individual classrooms. The cost of these programs was probably the only reason for a classroom in the state to individualize in any other manner.

By 1976 Harold Schoen had completed research that seemed to indicate that this type of individualization was no more a panacea than other methods. Although the research has its limitations, there were several findings that seem appropriate to report.

1. At the primary grade levels the findings appear to be ambiguous, but the results in grades five to eight very definitely favor the control (nonindividualized) teaching approaches (p. 91).
2. In grades five through eight the self-paced approach has very definitely not been effective in mathematics as measured by any criteria used by these researchers. In fact, it is rare that a question in educational research has such a clear-cut answer. The few studies supporting the individualized approaches are far outweighed by results favoring the traditional approaches even on affective measures (p. 94).

3. The educational quality of the pupil-teacher interaction in the self-paced classroom is very poor, consisting mostly of procedural matters (p. 93).

Schoen points out that teacher or principal should not feel that he or she is necessarily failing to allow for individual differences if he or she decides not to implement a self-paced instructional approach.

The self-paced method of individualizing instruction primarily varies the curriculum for individual students by manipulating the amount of time spent on a given topic in the curriculum. There are some authorities who feel that the variables associated with how students learn are more significant than the variable of time. A review of the literature will reveal other variables that are often considered to be significant. Many of these tend to emphasize learner style as one significant variable to consider in providing for individual differences in the classroom. Klingele (1979), in his discussion of instructional coordination, suggests categories such as tactile, auditory, verbal and combinations of these three. He lists implications for individualizing for each of these combinations.

In *Mathematics in Science and Society* several of the cognitive styles such as field independent or analytical as opposed to dependent or global are discussed. The authors illustrate that the global style students will have difficulty finding simple figures in complex drawings or pertinent data in a word problem, table or graph and they give some instructional implications for this aspect of learner style. The reflective and impulsive learner is also discussed. The authors discuss the middle grade students in the extremes of each of these characteristics such as the impulsive student who gives so many incorrect answers that his self-concept is at stake. The authors suggest some implications for combinations such as discovery lessons for the field independent reflective student as opposed to the field dependent, impulsive student who needs much more directed study.

Another significant factor often discussed in the literature is group size as it relates to interaction needs of students. Some students work extremely well when working alone. Some seem to be most productive in a small group while others do very well in a large group.

If Schoen's research casts some doubt on the effectiveness of manipulating only one variable in individualizing instruction, it becomes even more complex if other variables also need to be considered. What then are some other instructional methods of providing for individual differences? In 1977 the National Council of Teachers of Mathematics published a yearbook entitled *Organizing for Mathematics Instruction*. This yearbook attempted to give classroom teachers alternative methods of individualizing. The collection of essays submitted and accepted for the yearbook did not present all of the alternatives or a definitive treatment of any one of the organizational alternatives possible. A review of the chapters gives an indication of the solutions proposed.

1. Organizing for Individualization: the IGE Model
2. Organizing for Individualization: a Department Model
3. Organizing for Individualization: a Classroom Model
4. Organizing Independent Learning Units
5. Organizing Goal-referenced Instructional Units
6. Organizing a Learning Cooperative: Survival Groups
7. Organizing for Simulations
8. Organizing for Mastery Learning: A Group-based Approach
9. Organizing Instruction: Logical Considerations
10. Organizing for Alternative Schools: An Integrated Curriculum
11. The Teacher-centered Classroom
12. Implications of Research for Instruction in Self-paced Mathematics Classrooms
13. Hand-held Calculators: Past, Present and Future

A review of the chapters shows that some of the suggestions are concerned with grouping patterns, some curriculum and some teaching strategies. These suggestions could be helpful to the leadership of local school systems in systemwide planning in mathematics, as well as to individual teachers.

There are many administrative techniques that might facilitate dealing with individual differences in what students learn and how they learn. Some of the techniques include special classes in mathematics, departmentalization, flexible scheduling, team teaching, homogeneous groupings and mathematics laboratories. Many times, however, the teacher has a section of 25 to 35 students that have widely varying academic abilities and learning styles. The suggestions made in the remainder of this section are for those teachers of heterogeneous classes of approximately 25 to 35 middle grade students although they could certainly be used in classrooms or situations resulting from one or more of the above administrative techniques.

What Students Learn

Instruction for middle grade students certainly must include the concept of varying **what** students learn. If an individual classroom teacher is faced with the total responsibility for providing instruction for approximately 30 students in a heterogeneous setting, he/she will likely be planning for students whose mathematical achievement has at least a five year span and whose learning styles, rates, and the like are equally as diverse. The plans that the teacher develops for a given unit have to be plans that are reasonable to implement given the nature of his/her own personality, the nature of the students and the nature of the resources available. The extent to which a teacher can individualize mathematics instruction through varying the content for the students will be affected by all of the above. Most teachers realize it is unreasonable to expect the same content mastery from every individual in a class containing at least a five grade span in achievement but that it is equally unreasonable to attempt to provide meaningful instruction for 30 students with individual programs for each student. The suggestions presented for consideration in this section exist somewhere between every student assigned problems on the same textbook page and no two students on the same specific topic. The suggestions involve a combination of total class activities and intraclass groupings. Although the entire class would be working on the same topic, e.g., decimals, fractions or geometry, the groups would be pursuing the topic at various levels of difficulty.

To illustrate the concept of different content for different groups of students within a class, consider a unit on decimals for a seventh grade class. At the beginning of the unit when the class is looking at place value notation of decimal fractions, the group with a poor background in mathematics would be working with concepts such as $.34 = 34/100$. Another group might be writing out the meaning of 34 hundredths in expanded form such as ($.34 = 3/10 + 4/100$) or ($.34 = 3 \times 1/10 + 4 \times 1/100$). Another group might be using negative exponents in their expanded notation ($.34 = 3 \times 10^{1} + 4 \times 10^{2}$). Some advanced students might even be working on scientific notation as an extension of decimal notation ($.34 = 3.4 \times 10^{1}$). The entire class is dealing with the notion of the meaning of 34 hundredths but the level of difficulty for each group varies considerably.

In multiplying decimals, the groups would probably range from simple multiplication such as $42 \times .05$ in a problem context of figuring sales tax to $42 \times .13$ or 4.2×1.3 to problems such as $(4.5 \times 10^5) \times (3.6 \times 10^4)$. Each group is multiplying decimals but the content is certainly different for the various groups of students.

In writing the unit plans, the teacher might write different objectives for each group, i.e.

Group I - The students will be able to write a decimal fraction as a common fraction. ($.34 = 34/100$)

Group II - The student will be able to write a decimal fraction in expanded notation. ($.34 = 3 \times 1/10 + 4 \times 1/100$) or if desired ($.34 = 3 \times 10^{2}$)

Group III - The student will be able to write a decimal fraction in scientific notation. ($.34 = 3.4 \times 10^{1}$)

The teacher might prefer to write one general objective and then have different levels of expectations for the various groups. A general objective might be: the student will be able to explain the meaning of a decimal fraction such as thirty four hundredths. The objective might contain specific parameters such as tenths through ten thousandths.

No matter how the teacher chooses to organize the content for a heterogeneous class, it will likely be difficult to implement the plan successfully. The method of integrating intraclass groupings and total class instruction, however, seems to be one method that has potential for enhancing the instructional program in a heterogeneous class. Since the total class would be working on the same unit, some total class instruction could take place. Teaching via bulletin boards, learning centers or correlated enrichment activities, would also be possible. The group work, however, would break the content down so that there would be reasonable expectations for each student.

How Students Learn

Although most teachers would agree that learner style is a variable that is equally as significant as intellectual capability in providing for individual differences in a classroom, it is certainly more difficult to diagnose and prescribe appropriate activities for a diverse class of middle grade students. Not only do the individual students have vastly different learning styles that are difficult to diagnose, but the styles will likely vary from day to day because of the nature of the middle grades learner. A particular student may be a reflective learner one day and impulsive the next depending on his or her mood. Although individual students usually work best in a small group interaction mode, on given days they might not need this type of instructional format.

It certainly would seem that the teacher should be aware of these learner styles but it might not be wise to attempt to classify students nor prescribe activities on the basis of a classification. Rather it would seem a reasonable approach to plan a variety of activities for the total class or intra-class groups so that at best most of the students needs are met most of the time or that at least some of the needs are met some of the time. Certainly the mode of operation in some classrooms—checking homework, teacher explains new concepts and students work a few problems in class and finish remainder at home—done day after day for 180 days stands little chance of individualizing learner style to any degree at all.

In providing for individual differences in learner style, there are at least three factors that might be considered, i.e., materials, activities and interaction modes.

Materials and Activities

In providing for individual differences in a heterogeneous classroom, the teacher could use three different level textbooks if the system has a single textbook adoption. For example, for the seventh grade class studying decimals, the teacher might use a sixth, seventh, and eighth grade book for the various groups in the class. Since most textbook series provide assistance to the teacher for enrichment and remediation, this arrangement would provide for approximately a five year span. Multiple assignments could then be made for the students using the various books. Since the teacher would not want to call attention to those working below grade level, the assignments could be placed on the board by group name rather than by book level. Some groups might like to suggest their own name or the teacher might simply refer to those using the sixth grade book as Group 1, seventh grade book, Group 2 and eighth grade book as Group 3. In multiplying fractions a sample assignment from a current textbook series would be

Group 1 - pages 195-198
Group 2 - pages 104-106
Group 3 - pages 92- 93

This particular series of one publishing company has accompanying ditto masters on multiplication which the teacher could also cross reference and use effectively. Most major textbook companies not only provide additional materials but also provide information on correlating the materials. These materials can certainly be used as a foundation for the teacher who is planning to use a combination of total class and intraclass groupings in attempting to individualize mathematics instruction in a given classroom.

There are many other materials that have been developed commercially or can be developed by a classroom teacher to assist in individualizing via materials and activities. The choice of materials and activities is a difficult one but one that is very significant if one considers it a viable way to address the issue of varying mathematics instruction according to **how** students learn as well as **what** students learn.

Most lists of suggestions for varying the materials and activities in a classroom would include these.

1. Learning centers
2. Mathematical games
3. Learning activity packages (LAP)
4. Contracts
5. Projects
6. Bulletin Boards
7. Supplemental activities such as tasks and materials that might be available for individual students or small groups of students to complete their classwork or to substitute for their classwork. These are usually housed in shoeboxes, file folders, plastic bags, etc. If no materials are needed, they might be posted in the room or stored in a card file.

The choice of materials and activities will vary greatly from school to school and classroom to classroom. There are many considerations that would seem significant in determining which of these or other materials to use in a given classroom to assist in individualizing mathematics instruction for that particular set of students. For example:

1. Are the materials and/or activities related to the **objectives** for the unit of study?

2. Are they appropriate for the student, i.e.,
   a. Do they present the material on the **academic level** for the student using them?
   b. Does the student using them have the amount of **independence** required? Some students may be mature enough to handle contracts and some may not be able to do so. Some students can operate fairly independently with the instructions for a game and some can not. Some can handle small materials with care and others tend to throw them away or misuse them in some way.
   c. Are they compatible with the **nature** of the learner? Many middle grades students do very well with the LAP's since they are short mini units. This type of material is consistent with their enthusiastic but short lived interest in a particular subject. Some do well with learning centers because of their excess energy and their need to move around while some cannot assume the responsibility associated with learning centers.
   d. Are the materials and activities **relevant** to the student? Many times the additional materials and activities can be built around student interest. Some can be interdisciplinary in nature and consequently be more relevant to students.

3. Do the activities and materials for the unit collectively provide for a **variety** of learning styles? Are there some activities for the student who is a very poor reader, for the student who is very visually oriented, for the very bright student, for the student of average academic ability but very independent and so on? It is nice to plan some of the extra materials and activities that will be available to all students. This is particularly true of the supplementary activities which are often not available to the slow, deliberate, and conscientious students because they never complete their regular assignments.

4. Are the activities and materials that are used to assist in individualizing instruction **appropriate** for the **teacher**? This is probably the most important consideration of all. Teachers should feel comfortable with this consideration. Teachers have varied personalities and teaching styles just as students have varied learning styles. Some materials and activities are excellent when utilized by one teacher and a failure when used by another. It should be left up to the teacher to plan the materials and activities that

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blend with their own personalities. One example of this would be the use of mathematics games. Some teachers feel very uncomfortable with groups of children all over the room playing mathematics games but respond very well to a game that involves the entire class. There are many ways to effectively individualize through a total class activity. Almost any of the standard games can be individualized by color coding the activity cards given to various groups of students. For example, using a decimal activity, if the caller calls "95 hundredths," a card given to the slower student might show .95 or 95/100, a card given to average students might contain a section showing $9/10 + 5/100$ for him to cover up, and a card given to a more advanced student might show $9 \times 10^{-1} + 5 \times 10^{-2}$ in the section to be covered up. The usual Bingo rules could be played. If teachers are free to select materials and activities that are appropriate for their teaching style, they should then assume the responsibility of selecting, modifying or preparing materials and activities that will reflect an attempt to individualize instruction according to learner style as well as academic ability.

Interaction Modes

In considering how students learn, the last variable for discussion is that of interaction mode. One of the major criticisms of some of the original individualized programs where students used a sequence of learning packets or units and moved through the material at their own pace was that in most instances students were expected to operate in only one interaction mode. They worked on their materials alone and interacted only with the teacher on a one-to-one basis. For some students this type of interaction mode was extremely effective. There are some students, however, who make greater strides in mathematics when they are stimulated by interactions with other students either in small groups or in class discussion led by a teacher who facilitates the interaction by asking thought provoking questions and utilizing one student's observations to stimulate another student. The current emphasis on critical thinking and problem-solving in mathematics almost mandates that students have a variety of interaction modes. The social/emotional needs of middle grades students would tend to lend support to the argument that students need at least some experiences interacting in various group sizes.

In addition to group size, the type of groups that are planned for students would seem to be a factor in the overall discussion of interaction modes. Consideration should be given to types of intraclass groupings other than academic ability. Many students have experienced growth in mathematics by participating in intraclass groupings based on student interest, student learning style or various tasks. If students have opportunities to participate in groups of this description, there should be less stigma attached to the ability groups. It should lessen the situations where students perceive themselves to always be in the "dumb" group. Since learning to work together on a common task is also an important aspect of small group interaction, using these various types of groups as well as the ability groups should provide yet further experiences in this area.

In addition to total class activities and various types of small group activities, there certainly should be some experiences planned for students where they are working alone or independently. Even though this might not be the most effective interaction mode for some students, almost all students need and enjoy some activities of this nature. Ultimately, much of a student's learning must be done on an individual basis so it is important that opportunities be provided in this area.

The most difficult aspect of implementing a program to incorporate many of these suggestions is developing an appropriate management system that suits the teacher, students and curriculum. One viable management system would be one that includes the concept of unit planning. Within the unit, after the objectives for the class or groups within the class have been selected, the teacher would probably begin to identify a large set of materials and activities to match these objectives. If there is a basic textbook series, the first step would probably be cross referencing the different grade levels of the text that the teacher has issued in his/her class. A list of all these materials as well as ideas for activities can be stored in a unit notebook which also contains individual lesson plans. Since the notebook would also contain copies of quizzes and tests, it would be easy for teachers to then evaluate their unit to determine if they have made some progress towards individualizing instruction for the individual members of their classroom.

Although no one middle grades teacher can plan instruction on a daily basis to accomodate students' varying levels of achievement, rates and learning styles, the unit plan is one practical way of attempting to provide more individualization in how students learn as well as what students learn. The development of an exciting unit in mathematics for middle grade learners, in order to include these attempts at individualizing, is a time consuming task. Many times teachers at the same grade level share the responsibility. This may be accomplished by working together on one unit or sharing units that are developed individually. For example, one teacher may develop a unit on decimals and another teacher may develop a unit on geometry. Even though no two classes are alike, the units usually contain a variety of activities and can be easily adapted for several classes.

In summary, if individualization is interpreted in a broad sense to include meeting all the needs of individual students in the class, many factors must be considered. These factors should include considerations of how students learn as well as what they can and should learn. Since a given teacher with 30 students and a 60-minute block of time allocated to mathematics can only spend one to two minutes per individual student per day, it seems important to develop some techniques for individualizing within the structure of the total class by using a combination of total class instruction, various types of intraclass groupings and individual student work. These techniques will involve the use of a variety of materials and activities. One method of organizing mathematics instruction to incorporate different academic levels and learning styles is unit planning. The extent to which teachers can manipulate all of these variables will largely determine the extent to which they will be able to meet the individual needs of the students in their classrooms.

Special Considerations

Using Technology

Middle grade students are growing up in a highly scientific, technological world. Nearly every facet of their lives is affected by technology. One obvious influence on these students is television. As teachers and schools compete for students' attention, use of technological equipment can be advantageous. Communication is an important aspect of everyone's education and advances in technology can assist the mathematics instructor as well as all teachers both in communicating to students and in building students' own communication skills.

Multimedia. Periodic use of equipment as familiar as the overhead projector can enhance teaching and learning experiences. Slide-tape presentations as well as filmstrips, films and video-tapes can help teachers vary their teaching strategies for their individual students with different learning styles. Materials are available commericially and can be teacher-made or student-made. Slides, for example, are easy to learn to make and students can be encouraged to produce such products as slide-tape presentations for different groups, e.g., students at other grade levels in the school or various groups in the community.

Producing their own videotape on a particular lesson or unit will help students not only to understand the material, but will help them in building their awareness of communication skills which are as important to mathematics as to any other area of study. For assistance in locating support for learning how to make slides and videotapes, obtaining materials and selecting materials see "*Planning for Instruction*—Using Support Systems" in this guide.

Calculators and computers. Obvious tools of the technological world available to students and teachers which seem even more specific to mathematics are calculators and computers. Why should experience with computers and calculators be included in a mathematics curriculum? How should middle grades mathematics teaching and learning involve such tools?

Why use computers and calculators in mathematics instruction? Mathematics teachers have explored possible uses of calculators and computers in the learning of mathematics for over two decades. Many interesting ideas have developed. The reasons that teachers might include one or both of these tools in the classroom are the following.

1. Motivating—Students usually enjoy, indeed prefer, working on mathematical tasks by using a computer or calculator. They are often curious and excited by the idea of getting a machine to help solve mathematical problems or complete exercises. With the vast array of electronic games and home calculators students are often more comfortable than adults with such microprocessors.
2. Problem solving—When the focus is upon learning to solve a problem, it may be helpful to reduce or eliminate the computational demands. If students are encouraged to focus on analyzing the problem and on developing a method of solution rather than on computing the results, they may be more willing and able to learn to solve challenging problems. When the activity involves applications which include numbers obtained from actual measurements, students may feel less bogged down with calculations if they are permitted to use a calculator or computer.

3. Algorithmic learning—Students can be encouraged to build their own procedures (algorithms). Though many teachers will still wish to emphasize a standard or conventional procedure, it may be valuable to ask students to explore their own alternative procedures on a calculator or computer. For example, how might a calculator be used to find a quotient without using the divide key? Or a student might write, test and refine a computer program for simplifying fractions. In such activities students may learn **about** algorithms (i.e., algorithms are built by people who can test, revise, and improve their creation).
4. Practicing—After fundamental meanings have been developed for a new idea, students often benefit from practicing with specific examples. Computers and calculators can be used by students to practice. Computer programs which offer a practice session for a student often feature the following: (a) tasks individually tailored to each child, (b) student responses required, (c) immediate evaluation of student response followed by feedback to the child, (d) repeated trials offered when errors occur, (e) encouraging reinforcers given and (f) performance records stored and reported to the student and teacher.
5. Applying and reinforcing—Many teachers assign computer/calculator tasks which are chosen to reinforce the ideas developed in a lesson. If a student applies his or her newly constructed knowledge, these ideas may become more meaningful and better remembered. For example, students may be introduced to a definition for addition of two fractions. To reinforce this definition, students might complete a computer program to find the sum of two fractions.

		Program notes
10	INPUT "FIRST FRACTION"; A, B	Numerator and denominator are typed separately for each fraction (e.g., 2, 3 for 2/3)
20	INPUT "SECOND FRACTION"; C, D	
30	PRINT A; "/ "; B; " + "; C; "/ "; D; " = ";	The PRINT instruction types the addition phrase (e.g., 2/3 + 4/5 =). The blank could be completed with (A*D + C*B), "/" B*D
40	END	

By running the program and checking the output (which involves student practicing, too), the student's concept is further tested and reinforced.

6. Demonstrating and experimenting—Because numerous examples can be calculated and displayed easily and quickly, computers/calculators can be effective aids to demonstrating (and exploring) various instances. For example, students might find and record 1 ÷ 2, 1 ÷ 3, 1 ÷ 4, . . ., 1 ÷ 20 using a calculator. After examining the list of decimals produced, various patterns might be noticed and described. Conjectures about these patterns could be elicited from the students. For example, the number of digits in the part which repeats for decimals of the prime number denominators appears to be one less than the prime each time. Such conjectures could then be tested with new instances. A different example of demonstrating/experimenting might involve a computer program used by the teacher. For example, the program presents a picture of a region which can be subdivided according to values input by the class. For a value of 16 the computer shows a rectangular region being cut up into 16 congruent parts. Then an input of 5 shows five parts with a different color and the fraction 5/16. The computer program can accept many different such inputs, drawing in each case a correct area representation for a fraction and printing the fraction name on the screen. Through the use of such a program the teacher is able to demonstrate (with an exploratory emphasis) models of fractions for the students.

The possible forms and extents of microprocessor technology are being explored in many Georgia middle schools. Perhaps the attitude which middle school mathematics teachers need to adopt is an open, questioning but receptive approach. Technology has great potential to enrich, extend and support effective mathematics teaching and learning. It will be important for each teacher to construct understandings for, and predispositions toward, using such technology.

How to use microcomputers and calculators for mathematics learning

During the past 20 years there has been a variety of computers and calculators used in mathematics instruction. These uses include programming, practicing, tutoring, gaming, simulating, testing and managing. A thorough understanding of them will go beyond the scope of the current guide, but a brief characterization and illustration of possible uses of these microprocessors in programming follow.

Mathematics students of all ages throughout the world are writing and executing their own computer/calculator programs. When a student builds a program, the emphasis can be upon valuable outcomes of instruction. A learned noncomputer/calculator procedure might be reinforced as the student analyzes the steps in order to **tell** the machine how to do it. The teacher may begin by demonstrating on the chalkboard or overhead projector the steps of a procedure, such as finding the greatest common factor. Several specific number pairs are tried as the teacher models a procedure. Then students might be given an incomplete program and asked to complete and execute their program, testing the output to check their program.

		Program notes
10	REM FINDING GCF OF 15 AND 24	A REMark helps label a program.
20	FOR D = ___ TO 1 STEP -1	15. Sets up a loop of possible divisor 15, 14, 13, . . . 2,1.
30	IF 15/D = INT (___) THEN ___	15/D. Test for even division by comparing quotient with integer part of the quotient.
40	NEXT D	For values of D which are factors of 15, jumps to line 50. Otherwise, next D.
50	IF 24/___ (___) THEN ___	D. 24/D. 40. If D doesn't divide 24, jumps to 40. If D is a factor of 24, goes to line 60.
60	PRINT "GCF OF 15 AND 24 IS"; ___	Line 60 types the answer, D, the largest value which divides both 15 and 24.
70	END	

A student would need to reason how the computer could perform the steps in order to find the GCF of 15 and 24. Usually students will think about the computer actions, tracing through step-by-step the flow of events. To do this, the student must be aware at some level of operation how each instruction in the computer programming language is performed during program execution. Natural computer programming languages, such as BASIC, make it relatively easy for beginners to understand a program. Often a wide variety of possible programs can result from any algorithmic task set by the teacher. This requires that the teacher accept solutions which are different. Of course, each program can be tested for correctness by running it on the computer. The significant idea is that a student has constructed his or her own algorithm which incorporates the child's meanings and reasonings.

Some implications. Along with the introduction of the computer and calculator into students' lives should come an increased emphasis on teaching estimation skills. Formerly tedious calculations can now be eliminated by using the hand-held calculator, but people must know if their answer on the display is "in the ball park." Errors in entering data or operations, weak batteries and various other phenemena sometimes cause answers on a computer or calculator to be wrong. Humans must be able to make decisions with respect to appropriateness of answers. Middle grades students need skill in efficient, quick calculating for estimating. Rounding off skills, placement of decimal points and simple computations should be stressed.

Processes for problem solving become more important in middle grades students' lives. There will be more opportunities for students to set up problems and select which operations to use as they spend less time on tedious computations.

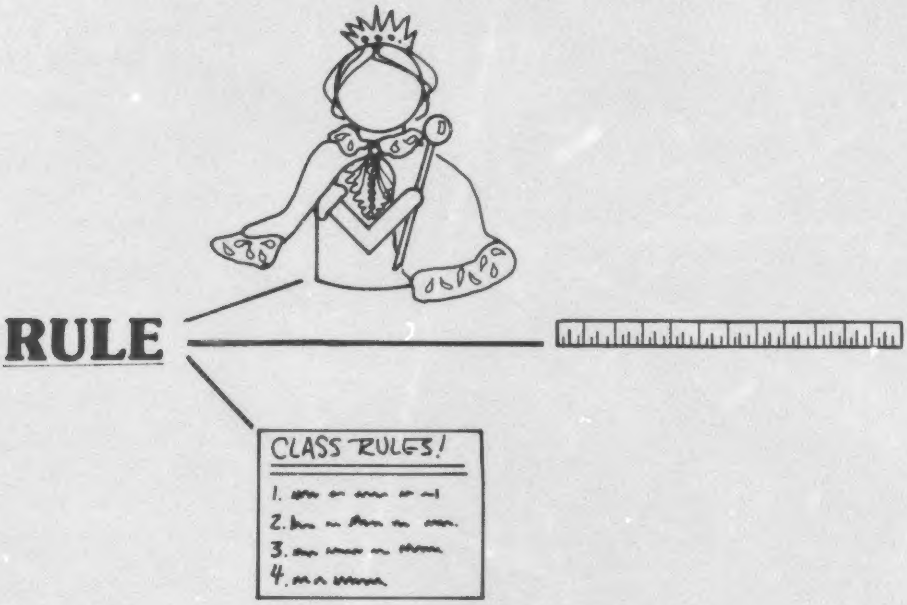
Technology can be used effectively to help teachers and students learn in a modern world. Flexibility and a willingness to try new things will inevitably help create an atmosphere where both the teacher and students can grow in an understanding of mathematics in a changing world.

Reading for Problem Solving

Many reading factors affect students' abilities to function successfully in mathematics. These factors naturally affect the students' abilities to solve word problems more than any other area of the mathematics program, but overall performance in mathematics is also dependent upon these factors. There are many similarities between reading skills necessary in a school reading program and those necessary in mathematics. Consequently, most students who experience difficulties in their reading program are likely to have difficulty in areas of mathematics that are dependent on the same skills. There are also important differences which suggest a need for increased emphasis on those which are specific to mathematics. Need for these additional skills makes it more likely that students with very few reading problems in general might have difficulties in mathematics because they lack these additional reading skills. When these students encounter difficulties in mathematics, consideration should be given to whether or not they do in fact possess those reading skills specific to mathematics. A few of the factors and some possible solutions are presented in this section to assist teachers in providing quality instruction in mathematics.

Vocabulary

The vocabulary of mathematics, one factor which can cause student difficulty, can be categorized into three classes; technical, specialized and general. Technical vocabulary consists of numerals and those symbols and words which are specifically mathematical, e.g., 5, 67, 2000, $\sqrt{}$, –, ¢, millimeter and decimal.



Technical vocabulary may be alphabetic or nonalphabetic. Specialized vocabulary may consist of words found in general use outside mathematics, which take on other connotations, more specific meanings or extended meanings in a mathematical context. Instances in the class of specialized vocabulary include product, difference, improper, yard and axes. General vocabulary is made up of all the words used in mathematics. These words are probably in the reading vocabulary of the students but their comprehension may be hindered by the unfamiliar context, e.g., and, or, if, then and only one (Pachtman, 1978 and Earle, 1977).

Many of the words in the technical vocabulary have no meaning for students until they are taught. These can be taught in the usual ways such as through the use of context clues, graphic clues or word structure. The teachers and students need to be aware that there is a mathematics vocabulary that must be developed in the same way that other vocabulary is developed. The other type of technical vocabulary, symbolism, poses a problem for many students in mathematics. There may be several reasons for this difficulty. First of all, there are no *clues* such as those used to build word attack skills. Another reason is that some symbols have several meanings. For example, the symbol "–" might be used in expressions like $8-2$, $^{-}3$, $\frac{10}{5}$ to indicate subtraction, the opposite of, and divided by. These symbols are to be taught as translations or representations of ideas.

Words that are already a part of the students' vocabulary but which have different meanings in a mathematical context may cause more difficulty than the technical vocabulary. When given a word association activity **in a mathematics class,** students gave the following responses: volume—radio, prime—beef, ruler—king, plane—fly, yard—grass, power—king. Therefore, words such as these used in mathematics class could be confusing to the students if they think of the meanings of the words as they are used in everyday experiences rather than the mathematical context. To minimize this difficulty, the teacher might have the students list words that have different meanings in mathematics than in everyday experience and discuss them in both contexts. The students could prepare a bulletin board or other demonstrations to illustrate both contexts. This activity could be expanded to include words that have the same meaning as, or at least can be associated with, the root word. For example, perimeter of a polygon can be associated with a perimeter highway which may be familar to students. Another example, the associative property used such as $(3+4) + 5 = 3 + (4+5)$, students can understand that the numeral 4 can choose to associate with its friend 3 or its friend 5. The students may also be familiar with commuting to and from work which may assist them in understanding the word *commutative*.

Fluency

In addition to vocabulary, the fluency with which students attempt to read mathematics may cause difficulty.

Students are often encouraged to increase their reading speed and, in the process, may develop the misconception that a fast reader in mathematics is a good mathematician. Fluent readers may, however, encounter difficulties in mathematics due to their reading speed. The fluent reader does not attend to each word in a prose context in order to comprehend. Mathematical problems are written in brief and condensed language which involves special meanings for otherwise familiar words, unfamiliar words and/or symbols. Readers of mathematics must therefore slow their pace and attend more closely to the words and symbols. Rereading is often necessary in mathematics. Several suggestions for bringing students' attention to the fluency of reading are these.

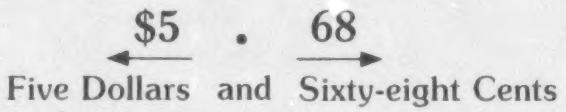
1. Choose a selection containing compact symbolism. Tell children to translate the selection into words. Compare the lengths of both forms of the selection. Ask why it often takes longer to read a selection in the language of mathematics.
2. Choose a selection containing a question to be answered. Allow children to read the selection once. Ask them to answer the question. Can they do so? If they can, why? If they cannot, what do they need to reread? Help them find the necessary information.
3. Pair children with one another. Assign a selection to be read—silently at first. Next, have the children read the selection aloud. Assign a selection of equal length from a reading book. Repeat the experiment of reading silently and aloud. Which selection required more time to prepare and read? Why does one selection require more time than the other?
4. Before beginning a chapter, and upon finishing the same chapter, have children write a three sentence summary of it. The summary based on a detailed reading of the chapter should show greater comprehension of the contents than the summary based on a preliminary skimming of the chapter.

(Kane, Byrne and Hater
1974, pp. 41, 42)

Directionality

A third feature in reading that might cause some difficulty in mathematics is directionality. Reading of prose or poetic text is essentially a left to right, top to bottom activity. Frank Smith (1978) has indicated in *Understanding Reading* that there are right to left regressions and bottom to top regressions in reading, but it does remain for the most part a left to right, top to bottom task. Mathematics, on the other hand, contains a variety of directional modes. Within word problems, directionality does not become a factor of major difficulty until students begins to analyze the relationships between the information given and the applications of computational knowledge which must be made. The first information which must be utilized is generally found in the last statement of the problem. The final information relayed to the student is most likely the purpose for which all the other information is given, that is, what they are asked to discover is the last thing they read. Depending on the problem, the students must back up through the information relating what is given to the stated purpose, or they must take that purpose given in the final statement and begin reading the problem again in the light of the purpose they have discovered. This configuration can cause difficulty for students who are accustomed to predicting and comprehending on the basis of only one reading of the material.

Expressions such as $5.68 are often found in mathematics. The directionality in such expressions requires much more than a left to right reading. The reading here reverses the symbols $ and 5 and requires returning to the decimal point for the concept of cents. Another instance of change in directionality is within the statement that there is a 4/5 probability that a blue ball will be chosen. When the students reach the fraction they must desert the left-to-right direction and adopt a top-to-bottom direction. Then they must resume the left-to-right mode for further information. Although the problem of directionality will be most severe in the earlier grades, there may be some middle grade students who are still experiencing this difficulty. It might be helpful to discuss this concept with the students and let them chart the direction. This activity will reinforce the necessity for reading mathematics more slowly. More specific examples of this skill and suggestions for developing it are found in *Reading Mathematics*, a publication of the Georgia Department of Education.



Solving Word Problems

All of the above considerations of reading factors combine to make the solving of word problems a source of frustration for most middle grade students. The students are called upon to use their skills in understanding vocabulary, adapting to unusual eye movements and dealing with adjusting their fluency to interpret the various pieces of information contained in a word problem. In addition, they must picture relationships in regard to information given in the problem. Richard Earle (1977) lists perceiving symbols, attaching literal meaning, analyzing relationships and finding solutions as the reading steps involved in word problems. In analyzing relationships the students must employ the skills of critical thinking, even though many of these relationships are unstated. Further, the students must draw logical conclusions with which to work and arrive at an appropriate solution (Ley, 1979). After all these processes have been accomplished correctly, all that remains is the mathematical calculation and checking the results with the problem to see if they indeed answer the question(s) posed.

Perception of the relationship of application to computation is the step in reading mathematical problems that appears to be crucial to the solution of those problems (Ley, 1979). Reading a given problem at this point must serve several purposes. First, the purpose for reading or comprehending is to make the question asked comprehensible to the reader. As has been noted previously, this task has been made more difficult by positioning the question at or near the end of the problem. Second, the readers must interpret the information given in light of the purpose they have determined. They must filter out information which is irrelevant to solving the problem. This process is especially important when it applies to information containing numerical designations. The readers must then establish relationships among all the pertinent information. These relationships must be inferred by the readers because usually they are not stated in the problem. Students may feel frustrated if they have not been led through successful experiences in making relationships in the past. At this point in their reading, the students must reach further into the comprehension experience that is required in literal comprehension of the symbol and word vocabulary. Finally, the readers must translate the relationships they have discovered into mathematical operations and their symbols.

Although students approach solving a problem in various ways — and this diversity should be encouraged — steps can be formulated to provide a foundation for students to use in solving word problems. Of course the steps become more critical as the problems become more complex and some students will not need to use every step. Some of the steps follow.

1. Locate the question
2. Find the given information
3. Sort out the relevant and irrelevant information.
4. Find and state one or more relations
5. Translate the relation(s) into mathematical symbols
6. Compute to find the solution
7. Use the solution to answer the question
8. Check to see if your answer is reasonable

Although steps 4 and 5 will likely be the steps where most middle grades students have difficulties, many will experience difficulties with other steps. The following four suggestions are provided to assist the teacher in working with the students on solving word problems.

1. Allow children to use varied techniques to solve problems. Have materials available to help children see concrete examples of the problems to be solved.
2. Whenever possible, introduce a new lesson by posing a problem to be solved. Allow children to use their own methods. Then complete the lesson development by dealing with the concepts involved and showing how various approaches lead to the same answer, and how this equivalence may then be expressed using appropriate symbolism. To find the area of a 3 by 3 inch square, children can be led to see the equivalence of the following computations and the symbols used to express them.

1	1	1
1	1	1
1	1	1

$1+1+1+1+1+1+1+1+1$	$=$	3×3	$=$	3^2	$= 9$
counting squares (addition)		multiplication		expressing area using exponents	

3. When a problem is to be solved, ask questions like the following.
   What are you to find?
   What information is given?
   Do you need all the information?
   Is needed information missing?
   How could you solve the problem?
   Could you solve it another way?

4. Have children describe to each other the methods used to solve problems. Help them see how various methods may lead to the same answer. Ask them to decide what they think to be the best method of solution from the point of view of saving time and using the least number of symbols. Then ask them to decide what they think to be the best method for determining a solution from the point of view of understanding the solution.

(Kane, Byrne and
Hater, 1974, pp 66)

Notice that these suggestions include activities for the entire class or at least groups within a class to solve problems with the teacher leading the discussion. The choice of interaction mode and the amount of teacher direction will vary as students become more independent in their ability to solve word problems. It is very important when working with students who are unsuccessful solving word problems in mathematics for the teacher to be aware of growth in the affective domain as well as growth in the cognitive domain. Many students have a negative attitude about solving word problems and as a consequence do not spend enough time **trying** to solve them.

The following plan of action demonstrates one way in which the affective domain, vocabulary development and critical thinking can be meshed into a program to assist students to become more successful solving word problems.

Stage One. It will be essential to begin this plan with special emphasis on the attitudes and feelings of the students. Towards the enhancement of the positive aspects of these feelings and attitudes, it is suggested that work be done with small groups of students who are having similar difficulties. A one-to-one situation may be necessary for students who have had many failure experiences or who have poor self-concept. By working in the small group it will be easier for the teacher to control attention, response and organization in the students. Teacher interaction must be present in this stage but it should not become teacher intervention or teacher dictation. Assistance should be the criteria for teacher action, **initiative** should be the desired behavior for students. In this stage the teacher would work closely with the group by giving several word problems and then working with the group as they solve the problem together.

The students would work through a set of steps as a group identifying the problem, locating the information needed to solve the problem, finding irrelevant information and on through the various steps.

Stage Two. The material used in this stage should be presented to students who are working in pairs. They are to work together and come to concensus on acceptable solutions to the problems. The students should be aware that they may obtain assistance of the teacher only after they have attempted to come to this concensus but have not been able to do so. Care should be taken that none of the problems presented contain mathematical computations with which the students are unsuccessful.

Stage Three. At this level the students should pursue the material on their own. Teacher intervention should occur only when a student becomes frustrated and unable to function with the material. This final stage should be carried on in an individual setting. This setting may be as formal as a learning center or as casual as working alone at a desk on the materials. The students should not be put on their own until they have enough successful reading experiences to make this a comfortable experience.

One of the primary objectives of this program is to make the students aware of the difficulties and differences they may encounter in mathematical word problems. Reading competence does not insure mathematical competence because of the variety of demands mathematics can make on reading comprehension. The students can be led to discover adjustments they need to make in their reading to achieve succcessful results. Another objective is to develop and enhance the critical reading and thinking powers of the students. Since the analysis of relationships is crucial to the solution of mathematical word problems, it is important that these critical reading skills be given attention. Assumptions about vocabulary competence need to be explored by the teacher. It is not possible for students to form the proper relationships if they are not using the vocabulary correctly in context.

Being able to solve word problems is, after all, crucial in terms of **basic skills** for every informed citizen. Problems that students encounter in their present lives and in the future are seldom found already stated in mathematical expression. Problems are more likely to present themselves in the form of a situation which the students must interpret and in effect, create "word problems" of their own to solve. Both in the students' future courses in mathematics and in their everyday lives they will need skills in solving word problems successfully. The importance of these skills cannot be over-emphasized.

In summary, it is evident in the school situation that although reading and mathematics require similar skills, there are differences which cause difficulties. These difficulties account for some children being competent in reading but unsuccesful in reading mathematics. This situation is particularly true in the area of reading and solving word problems. Vocabulary building is basic to any plan for improving reading in mathematics. Attention should be given to the technical vocabulary in mathematics including symbols, specialized vocabulary, which includes words already in the everyday experiences of students but with special meanings in mathematics as well as general vocabulary used in mathematics context. Assumptions about vocabulary competence need to be explored by the teacher. It is impossible for students to form proper relationships if they are not using the vocabulary correctly in the context. Directionality and fluency are also factors that need to be considered. The students should be guided to discover adjustments they need to make in these areas of their reading to achieve successful results. Solving word problems provides mathematical experiences that require proficiency in all of these skills. In addition to these skills, the information must be analyzed carefully and relations formed and solved in order for the solution to the word problems to emerge. Any procedure utilized by teachers to provide successful experiences for their students in reading mathematics should culminate in students' achieving a greater degree of independence in reading mathematics and solving word problems.

Underlying the total effort is the teacher's awareness and consideration of the aspects of the affective domain. Many mathematics students do not do well because of negative attitudes and feelings within themselves. These attitudes can be overcome by the patience and cooperation of a sensitive teacher and students who come to see the relevance and value of the work. Successful experiences in reading mathematics and solving word problems are keys to positive attitudes and self-confidence.

Mathematics Anxiety

The most stressful content area of the middle grades for many students is mathematics. This stress is often times demonstrated in fearful feelings, sickness, hostility, anger or restlessness. Mathematics anxiety can be the source of discipline problems as well as academic problems. The following scenarios are provided to assist in examining some of the causes and possible solutions to this anxiety.

Possible Causes

Readiness. Richard is a bright and creative student in the seventh grade. He is an avid reader and has made excellent grades in reading, language arts and social studies. Because of his overall performance in school, Richard has always been expected by his teachers and parents to perform in a similar manner in mathematics even though his aptitude in mathematics is somewhat lower than his verbal aptitude. His experiences in mathematics, therefore, have always been inappropriate for his level of development or achievement. He has always struggled to keep up in mathematics and even though he has average ability in mathematics, his anxiety has compounded each year. Consequently, by the seventh grade, Richard experiences **serious** learning difficulties.

Language. Larry seems to have the potential to be a good student in mathematics although he is having many problems in reading and those areas of the curriculum that are extremely dependent on reading skills. He is very responsive in group discussions in mathematics classes but is highly frustrated when he is assigned work in his textbook. He refuses to even attempt any word problems. He is even more frustrated because mathematics was his best subject in the very early grades.

Motivation. Mary is not a typical math-anxious student but she constantly says that mathematics is dumb. She complains about working page after page in her mathematics book and is always asking why she has to learn things like dividing fractions. Her negative attitude is particularly harmful as new topics are introduced in the mathematics curriculum. This attitude and lack of motivation make it difficult for her to learn these new concepts and thus she becomes anxious and contributes to the anxiety of other students in her class.

Home. Harry is a very happy student in school until the mathematics lesson begins. He becomes shy and withdrawn at that time. His self-concept is very poor, and he becomes frustrated very easily in regard to mathematics. A conference with his parents reveals that both of his parents had difficulties in studying mathematics, and they have convinced Harry that this destiny is his also.

Stereotyping. Sue is typical of many girls in her grade who are math-anxious students because of the sterotyping that is perpetuated by parents, teachers, and other students. Sue is a victim of the myth that girls do not need to know mathematics for success in their future lives. Also Sue must overcome the belief that it is not feminine to do well in mathematics.

Testing. Tracie is a student who does rather well in most of her classes and even in her mathematics class until it comes time to take a test. She can answer questions orally but freezes when given a paper and pencil test. Although testing is a problem for her in all of her subjects, it seems to be more of a problem in mathematics. This may be due to the frequency of tests in mathematics or the right or wrong nature of answers on a mathematics test. Failure in these testing situations increases the existing anxiety.

Whether a student's mathematics anxiety is similar to Richard's readiness difficulties, Larry's language problem, Mary's motivation or lack of motivation, Harry's home situation, Sue's stereotyping syndrome, Tracie's testing trials, or something entirely different, it is nevertheless very real to a middle grade student and a concern that needs to be considered in planning a mathematics program.

Mathematics anxiety is not confined to students of mathematics. There are many middle grades teachers who have anxiety about teaching mathematics. The uneasiness of many teachers in the area of mathematics may be attributed to a variety of sources. Many classroom teachers confronting a mathematics class have not been trained specifically as mathematics instructors. Some may have received training but have not had the opportunity to update their skills to accommodate the changing mathematics content and/or methodology. This lack of adequate preparation in the concepts and processes of understanding mathematics is a major cause of professional anxiety among middle grade teachers.

Many teachers may have had unpleasant experiences in mathematics classrooms during their own schooling. These bad memories can cause uncomfortable feelings toward the subject matter and this anxiety on the part of the teacher may transfer directly to the students.

Another source of anxiety may be created by the amount of formal testing that is done in mathematics classes. Although this testing is used to evaluate student's progress and mastery, many times teachers will use the student test results to evaluate their own performance. This single factor evaluation may not only be a misleading indicator of student achievement, but also not a conclusive nor encompassing instrument for the measurement of the competence of a mathematics teacher in the complex atmosphere of the middle grades. This erroneous assumption that low scores indicate an inability of the teacher can certainly contribute to the anxiety level of the middle grades teachers.

An additional source of anxiety may be a confusion in the roles and priorities the teacher is to address during the school year. Many teachers do not feel that they have the directions they need to meet the many-faceted demands of the modern mathematics classroom.

Mathematics anxiety, whether on the part of the student or the teacher, needs immediate and continuing attention. If it is allowed to go unchecked, it can fester until it becomes completely unmanageable.

Possible Solutions

After a mathematics anxiety problem has been established, one of the first steps in the treatment process involves an open discussion of the problem to assist in further diagnosing and attempting to find a solution. For the math-anxious student, this would involve an attempt to bring together the student, his or her parents, the teacher, and any other individual who could contribute to the analysis such as a school counselor, another teacher, *etc*. Once the problem is thoroughly diagnosed and discussed, a program should be planned with deliberate assessment made of the progress all along the way. For the students in the previous scenerios, there are additional sections of the guide that provide suggestions that might be helpful. For example, since Richard's problem was his own readiness at each level of the instructional process, the section on personalizing instruction should provide some assistance.

Larry's language problem is addressed in the section on reading in mathematics. Linguistic confusion seems to be prevalent in many students. Reading in mathematics is a specialized skill which receives little attention. Words have different definitions in the mathematics class. Reading is done in a variety of directions and symbols are used in a specialized way. Letters have new meanings. It is no wonder that the language of mathematics can cause uneasiness for students. The teacher can counter this confusion by spending some time on the particular features of the language which change within the concept being taught.

Teachers who wish to increase the enthusiasm and motivation of students like Mary have many sources of help. Middle grades teachers can try new channels of learning and new teaching environments. They may use transparencies, films, filmstrips, slides (commercial and home-made), and other audio-visual aids to demonstrate relationships and processes as well as algorithms. They can vary the seating arrangement of their room to encourage small group work or one to one instruction. Partners in learning can be formed by seating students in pairs and helping them to be responsible for each other. Materials can be gathered from many places to supplement the textbook. Many items such as cereal boxes and empty cans can be brought from home. Local business firms may be willing to donate a variety of items. Involving the students in decisions concerning the **how** if not the **what** of teaching can relieve many of the conflict areas in the classroom. Adjusting curriculum to the students can provide successful experiences for students and consequently both the students and teacher may feel more positive about mathematics and themselves. Additional suggestions for students like Mary are included in almost every section of this guide.

Harry's home situation (well-meaning parents who excuse poor performance or minimal effort in their children with the explanation that they also had problems with mathematics), is a difficult problem for classroom teachers. The building of good relationships with the parents can do much to improve the situation. It is important to help the parents to see that their attitudes can be limiting or destructive to their child. A parents' workshop in mathematics can create a greater understanding of the mathematics content and foster better attitudes toward the mathematics their child is learning.

Progress is constantly being made toward alleviating the stereotyping that has created Sue's anxiety. Girls should be given the opportunity to fulfill their potential in the mathematics classes just as they are in other classes. An awareness on the part of middle grades teachers can contribute greatly to reducing this source of anxiety.

Tracie's test taking anxiety may be reduced by introducing a variety of evaluation procedures rather than relying on a single factor mode of pencil and paper tests. Alternative evaluation methods to consider include discussion (class or small group), observation, projects, reports, classwork and homework. The solution to Tracie's problem may also reduce teacher anxiety related to the use of testing situations to evaluate a teacher's own competence. The other concern of Tracie's (*right or wrong* nature of answers) can be alleviated in many instances. For example, if asked the probability of a certain event, 3/6 may be as correct as 1/2 and this should be pointed out to students.

For the teacher, the solutions are not as obvious or as attainable. The primary help for the anxious teacher of mathematics in the middle grades must come from within. If the teacher can admit to his or her own insecurity and anxiety and recognize their existence and prevalence among middle grades teachers, it will be possible for that teacher to seek and accept help from a variety of sources. Fellow teachers can often be the first line of support for the teacher who feels the need of assistance in mathematics. Most teachers are pleased and eager to share their favorite activities and teaching methods for particular concepts and skills. The key to this type of support must be the readiness of the anxious teacher to ask for help and suggestions. It is then incumbent upon teachers who are more secure in mathematics to share appropriate methods and materials and to explain and demonstrate their use thoroughly. If a teacher charged with the responsibility of mathematics education is not able to gain sufficient or appropriate support from fellow teachers, it is possible to contact many other individuals to assist. The section on support systems in this guide suggests many other sources. Many times teachers may prefer to talk with someone outside their building about their mathematics anxiety because they are uneasy about the problem and someone they see daily might inhibit their progress. On the other hand, someone in the building would be available on a daily basis to support them, so the best source of assistance would have to be the determination of the individual teacher.

In summary, anxiety concerning mathematics is a significant factor in the lack of achievement for many students. This anxiety may be student-centered anxiety or the student may have difficulties that result from teacher anxiety in mathematics. Richard, Larry, Mary, Harry, Sue and Tracie represent only a few of the facets of this complex problem. The solutions to these problems are equally complex and require much open discussion with the affected parties. The other sections of this guide contain many suggestions that should be of assistance to the teacher once the anxiety itself and its causes have been determined. Although anxiety has developed over a long period of time and consequently cannot be corrected overnight, students' and teachers' feelings about mathematics can be changed and the potential for mathematics achievement enhanced greatly if a concerted attempt to alleviate the causes is begun.

Students with Special Difficulties

Rare is the student who has not at some point had difficulty learning mathematics. The severity and the persistence of such a difficulty varies. Individual characteristics play a part in shaping the breadth and depth of one's potential for academic achievement. Whether that potential is realized depends upon environmental factors including the school environment.

One individual characteristic is intelligence. This trait partially determines how far a student will progress in learning mathematics. It is difficult to specify limits for any person, yet intellectual limits do exist and may be a factor in a learning difficulty. Physical limitations such as perceptual or motor coordination problems, chemical imbalances, language difficulties or loss of vision or hearing can also be obstacles to learning mathematics. Emotional well-being likewise affects one's desire and ability to attend to instruction and might well be one source of a person's problems with mathematics.

Rare is the teacher of mathematics who has not experienced the frustration of helping students who have learning difficulties. Included in this guide is information about some aspects of mathematics instruction which can be modified to avoid or minimize difficulties many students experience in learning mathematics.

Modifying Instructional Approaches to Meet Learner Needs

Learning style is a term used to refer to a manner in which an individual takes in and processes information. Examples of characteristics of learning styles include inductive vs. deductive thinking, a visual-spatial vs. a verbal approach to taking in information and an impulsive vs. methodical manner in pursuing an idea or a task. Students vary with respect to learning styles, and each individual varies with respect to learning style for a given concept, depending on the material being learned and the student's level of cognitive development. Teachers who provide for variations in learner style increase chances for students to learn. (For more information see the section in this guide entitled, "Structures for Individualizing.")

A selection of variations in learner styles are addressed here. They are (1) the level of abstraction on which a mathematical concept or skill is learned, (2) modes of communication, (3) divergent and convergent thinking and (4) the amount of time needed to learn a mathematical concept or skill.

Level of abstraction at which a concept or skill is understood. Learners progress from the concrete level (manipulating physical objects which model a mathematical concept or process), to the pictorial level and finally to the abstract level (expressing mathematical ideas in symbolic notation). Instruction on the second or third level before the learner has observed the results of manipulating concrete materials might result in the student memorizing facts not based on understanding. The lack of understanding could cause a learning difficulty for the student.

Modes of communication. Some students learn best if the primary mode of receiving information is tactile, others if it is visual and others if it is auditory. Some individuals learn best through a given **combination** of these modes. For example, in demonstrating multiplication of fractions the teacher might cut an apple into halves and subsequently cut each half into fourths, producing eighths ($1/2 \times 1/4 = 1/8$). Another might prefer to use an eighth note in music being half as long as a quarter note. Instruction which is consistently provided in only one mode will be significantly less effective for the student whose strength for receiving and expressing ideas is in a different mode.

Individuals also vary in ways they pursue a solution to a problem. It is necessary at times to direct students through a logical sequence of ideas (convergent thinking); it is also necessary at times to allow them to explore ideas in their own ways (divergent thinking). For example, a scout troup is to sell 126 tickets for a fair. There are 9 children in their troup. How many tickets should each get if they are to recieve the same number. As students find how many nines are in 126, some may be ready to use the standard division algorithm while others may use repeated subtraction. A variety of methods to solve a particular problem should be encouraged.

Amount of time needed to learn a mathematical concept or skill. If a student is forced to move on to activities for learning new concepts or skills before enough time has been permitted for understanding the prerequisite concepts or skills, both prerequisites and subsequent ideas may be partially learned. If this type of situation happens repeatedly in mathematics instruction, the cumulative effect could cause the student to have significant learning difficulties.

All students need a mathematics instructional program which allows for manipulation of variables such as these (level of abstraction, modes of communication, thinking styles and time). Students with physical or psychological disabilities may suffer more than the other students if these variables are not manipulated. Who are these students, and what factors determine the extent of the effect of their disabilities on learning mathematics? Terms which are associated with disabling conditions include specific learning disabilities (SLD), behavior disordered (BD), educable mentally retarded (EMR), trainable mentally retarded (TMR), speech impaired (SI), hearing impaired (HI), visually impaired (VI), physically handicapped (PH) and other health impaired (OHI). The Education for All Handicapped Children Act of 1975 (PL 94-142) specifies that all handicapped children are to be provided educational services.

At least two factors influence the extent of the effect of a disabling condition on learning mathematics: (1) the **attitudes** of the student, teachers and peers toward the disabling condition and (2) the **extent** and/or combination of disabilities. The role of attitude is crucial to academic achievement. The extent to which a student's disability is a handicap in the pursuit of academic goals can be a function of how the student **perceives** the limiting effect of the disability. This factor is particularly significant when dealing with the preadolescent or adolescent student. During and preceding adolescence, peer approval and being like (rather than different from) one's peers takes on great importance. Having to deal with a disability which sets the student apart from peers sets the stage for an inner conflict. The extent to which disabled students can confront and resolve this conflict influences how open students are to the full use of their academic potential.

How classroom teachers of mathematics view disabled people and how they feel about the inclusion of these students in their classrooms have an effect on the disabled student's opportunities to develop in these teacher's classrooms. Increased knowledge about impairments, clarification of one's values regarding people with impairments and clarification of responsibilities of all involved parties in educational agencies (both state and local) are crucial considerations for the classroom teacher. (See Cole, 1977; Glass, 1973; Jones, 1977; Kavanagh, 1977; Lilly, 1975; Meyer, 1976; Roubinek, 1977; Wiederholt, 1977.)

Among the values to be considered by classroom teachers is the desire to change instructional approaches to meet disabled students' needs. Willingness alone is not sufficient for translating goals into reality. Many other factors, such as class size and the needs of other students might well limit how much teachers can do for disabled students. Accepting these students and learning about their specific needs will help teachers to do what is **possible** in given situations. Just as teachers have had to confront the issue of knowledge about impairments and attitudes toward disabled people, so they might also need to assist students to become sensitive to these issues. (See Aiello, 1977, for information about suggested approaches to meet this need.) It is in accepting environments that most human beings can best grow, academically or otherwise.

Just as **short people, tall people, heavy people,** and **slim people** come in different heights and weights, so too **disabled people** come with a multitude of differences. Some students diagnosed as learning disabled have visual perception difficulties, and yet have adopted compensating behavior so that their impairment results in little or no disability for learning mathematics. Other students' learning disabilities are compounded by additional difficulties such as memory deficits and problems with organizing information. This combination has a greater effect on learning disability. Some mentally retarded people have motor problems, while others do not. Some cerebral palsied people are mentally retarded, some are of normal intelligence and a small percentage of them are within the genius category. Thus being **learning disabled, mentally retarded** or **cerebral palsied** implies no guaranteed learning difficulty. Rather, the extent of the impairment and the physical or psychological abilities affected by that impairment play a role in determining how handicapped each individual is in the educational arena. (For an in-depth discussion of the effects of specific disabilities on learning mathematics and strategies for teaching students with specific difficulties, see Reisman and Kaufman, 1980.)

Seeking More Specific Information

A teacher should first gain a clarification of the responsibilites of the local school system personnel involved in delivering special services to students. If no special services have yet been provided for that student, it is the teacher's responsibility to (1) refer that student for assessment to determine whether a need for special services exists; (2) cooperate with school personnel in assessing needs and in designing the Individualized Educational Plan (IEP), which includes specific services to be delivered to that student and (3) implement and assess the effectiveness of that part of the IEP which pertains to instruction of subject matter for which the teacher is responsible.

In carrying out these responsibilities the classroom teacher of mathematics will need to turn to and cooperate with people in the local school system. Either the principal, the special educator or the building supervisor is responsible for providing answers to questions about the teacher's role in planning provisions for student services. These people can be particularly helpful in the initial stages of referral, in planning and in implementing IEPs.

To clarify the responsibilities of all involved school personnel, a teacher should ask questions such as the following.

- * Who is reponsible for writing IEPs?
  Each special education student must have an IEP designed to meet the identified needs. The Education For All Handicapped Children Act specifies that a person who knows the student's educational background must be present when that IEP is written. This person is usually the classroom teacher.
- * Who is responsible for determining whether objectives listed in the IEP have been attained?
- * Who will report academic progress to parents?
- * If no special educator or school nurse is in the building, what steps should be taken in an emergency situation?

  (For more ideas about defining the roles of the special educator and the classroom teacher in delivering services, see Roubinek, 1977.)

To gain a better understanding of the nature and the educational consequences of a student's impairment, a teacher might ask questions such as the following.

- * What is the nature, or if known, what is the cause of the disability? The information could be important for instructional planning.
- * Has the disability been present since birth? For example, if the student is blind, "the use of visualization in thinking (visual imagery) is absent in those born blind, and tends to disappear if sight is lost before five to seven years of age. Individuals who lose their vision later can usually retain a visual frame of reference; that is, they can feel an object and compare it mentally with their visual memory of other objects." (See Mordock, 1975, for more information.)
- * What are the student's most effective channels of receiving and communicating information?

- * Is there any behavior which **generally** characterizes individuals with this disability? In what kinds of situations is one more likely to see this behavior?
- * What are the student's attitude and the parents' attitudes about the disability, and what are the academic expectations for the student?
- * What is the interpretation and what are the educational implications of information contained in the assessment reports, e.g., those from audiologist, ophthamologist, psychologist? For example, a learning disabled student might need all test questions read aloud. If the student is scheduled to spend part of the day with a special resource person, perhaps this person could read the test to the student. Or the special educator could suggest alternative procedures.
- * What are alternative ways the classroom teacher could assess a particular student's learning?
- * Are there other resources from whom more information could be obtained? For example, an occupational therapist might provide information about a student who is orthopedically impaired.

If a teacher needs more information, the special educator, the principal or the school sytem special education consultant may be able to supply a list of appropriate agencies. Other possible sources include the local Cooperative Educational Services Agency and the Georgia Department of Education.

Planning and Instruction

Planning mathematics instruction for the disabled student requires that a teacher combine two areas of knowledge — first, the knowledge of the effects of a disability on a student's learning and second, a knowledge of mathematics and a variety of strategies for teaching mathematics. The integration of these two areas of knowledge is a complex process. Knowing more about each of these areas should help the teacher identify necessary modifications in the instructional environment and effective teaching strategies for each disabled student.

The process of identifying effective teaching strategies for a given **individual** will have to be one of experimentation. Guidelines for dealing with specific types of disabilities suggest characteristics of activities most likely to be appropriate for a student with those disabilities. Each individual within a given category of disability will vary with respect to degree of handicap as well as learning strengths and weaknesses; therefore, the classroom teacher must try a variety of teaching strategies to determine specifically what kinds of approaches best suit each individual student.

The disabled student differs in some ways from students in the normal population; likewise, the student shares similarities with them. Note that some variables of instruction (level of abstraction, modes of communication, divergent and convergent thinking) which need to be manipulated for disabled students are also the same variables to be manipulated for all students. No student with a disability needs a mathematics program entirely different from other students. A sound principle of teaching mathematics is that all students should be exposed to a concept in a variety of contexts and through the use of a variety of materials and examples.

Specifying Alternate Instructional Approaches for Disabled Students

Examples of teaching practices follow in order to give the teacher of mathematics ideas of **kinds** of instructional modifications students may need. The teacher who wants more help is encouraged to consult the special educator. The specialist's suggestions may pertain to **general** learning needs; therefore, the classroom teacher may need to apply these generalizations to learning mathematics.

Some disabled students have memory problems. They may forget oral directions very soon after having heard them; they may not differentiate between facts which need to be committed to short-term memory, e.g., the digits in a given four-digit number to be copied, and long-term memory, e.g., basic mathematics facts, or a multitude of stimuli may clutter memory centers in the brain. A memory problem often results in the inability to memorize basic mathematics facts. It is important to recognize that the plight of these students is **not** the result of laziness or poor motivation. One should **build on the learning strengths**

of these students. One learning disabled adolescent was observed determining the standard number name for 8×8 in the following manner:

$$8 \times 4 = 32 \text{ and } 32 + 32 = 64.$$

One interpretation of the mathematical properties applied in this procedure is that this student factored 8 as 2×4; $8 \times 8 = (2 \times 4) \times 8$; used the associative property: $(2 \times 4) \times 8 = 2 \times (4 \times 8)$; and interpreted multiplication as repeated addition: $2 \times 32 = 32 + 32$. Whereas a discussion of this student's work takes a few minutes, the computation was completed within seven seconds. Though this student was incapable of memorizing **all** of the basic multiplication facts, he had devised solution strategies which indicate sound mathematical thinking. This strength is his compensation for a particular weakness. Students who do not readily adopt satisfactory strategies benefit from instruction in which behavior like that of this student is explored. When computation becomes more involved, for example 0.5×36, the use of the calculator might well be the best strategy. Other students will want to compute. For example, one student mentally calculated 1/2 of $30 = 15$, 1/2 of $6 = 3$ and $15 + 3 = 18$. Not all students should have to apply identical procedures for answering a question. Teachers facilitate learning when they guide rather than discourage students who seek alternate solution strategies.

Teachers might consider adaptations of the **physical** activity of cutting an apple and displaying its fractional parts to model the product of 1/2 and 1/4. Suppose a hearing-impaired student and a sight-impaired student are to take part in this activity. Awareness of educational implications of their impairment should precede instructional planning.

Since the hearing-impaired student receives little or no information through auditory channels, visual and tactile stimulations are of utmost importance. This student will profit from engaging directly in the cutting activity. The teacher need not be with the student at all times to provide instruction; a student **buddy** can serve this purpose. Hearing-impaired students have trouble with words which communicate ideas about sequencing. To help this student recognize the sequence of events in this activity and to develop appropriate language descriptors, four pieces of construction paper, each of a different color could be used in the following manner. (See Figure P-1.) The teacher should print, "**First,** take **one** apple," at the top of the first piece of paper. (Underlining the word "first" and using color stresses sequence; underlining "one" stresses that the object is the **unit** of which a fractional part is to be found.) The student should place an apple on the paper. The teacher should print, "**Second,** cut the apple into **halves,**" at the top of the second piece of paper. The student should cut the apple on the second paper. The teacher should print, "Third, cut the

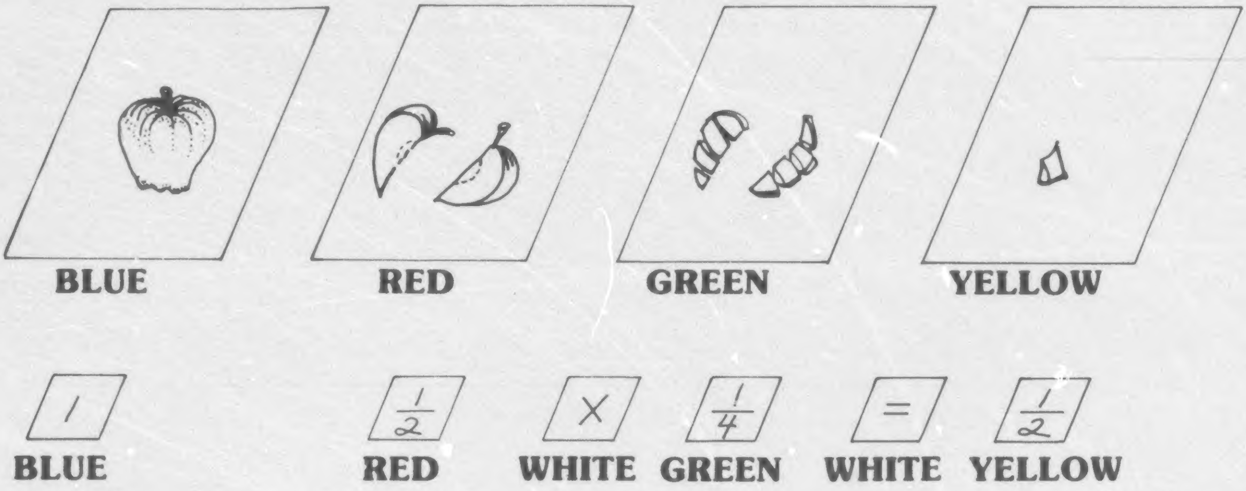


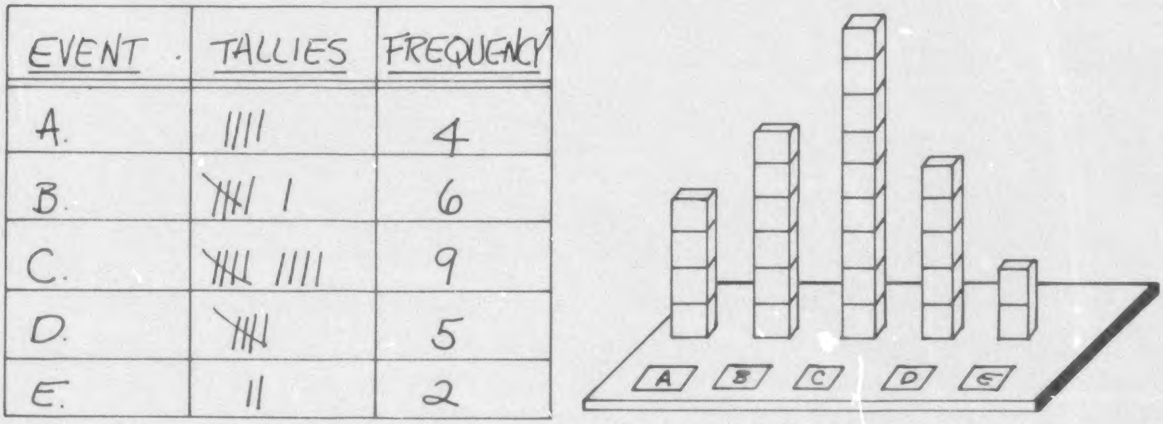
Figure P-1

each half into **fourths,**" at the top of the third piece of paper. The student should make the necessary cuts on that card. The teacher should write "**Last,** put a **fourth of one half** here," on the fourth card. To strengthen understanding of the use of mathematical symbols in communicating what has been done, the student should be given a 3" × 5" piece of construction paper on each of four colors used. The teacher should write, "1/2" on the card matching the color of the paper on which the apple was cut in half. The same should be done for the fourths and the eighth. The teacher should write the symbol "×" and "=" on white cards. The student should be directed to build the equation which matches the activity.

With a few modifications, this activity is appropriate for the sight-impaired student. Various textures of paper (plain, velour, corregated cardboard and paper with a slightly rough surface) could be used instead of using varied **colors** of paper. Directions might be typed in braille; the resource person in the school, or a local society for the blind should be able to help. Also, the Library for the Blind will do this typing for anyone in the state. Another student could work with this person and make initial cutting lines on the apple.

The hearing-impaired person can visually perceive the part-whole relationship of the resulting fractional part of the apple with the remaining parts which together constitute the **whole,** but the visually-impaired student does not have this advantage. After this student has completed the activity, another apple with approximately the size and shape of the one that has been cut into eighths should be provided. Lines should be cut into this apple to indicate where cutting would occur to partition it into eighths. The student should be directed to feel the difference in size between the whole apple and the one-eighth section of an apple and to feel the eight sections of the whole apple. This kind of experience — **feeling** the pieces with relative sizes of the part and the whole — should be repeated with different size and differently shaped unit objects and their fractional parts, to allow this student to generalize the part-whole relationship across a broad variety of examples.

In activities where tally charts are used, wooden blocks may be stacked to represent tallies if students can attend to the task rather than play with the blocks. (See Figure P-2.) Learning disabled, mentally retarded or orthopedically impaired students will probably find placement of blocks easier than writing on paper.



EVENT	TALLIES	FREQUENCY
A.	||||	4
B.	𝍸 |	6
C.	𝍸 ||||	9
D.	𝍸	5
E.	||	2

Chart

Table with Blocks

Figure P-2

The student with behavioral problems might find this procedure more interesting and might thus be more highly motivated. Feeling the varied heights of the columns, the visually impaired student has a tactile impression of the frequency of each outcome. The learning-disabled student with trouble understanding symbols written on paper might find this display easier to interpret.

In summary, suggestions have been offered here to assist the classroom teacher of mathematics in dealing effectively with some difficulties students encounter in the process of learning mathematics. All students are alike in that their learning strengths and weaknesses differ regardless of their position on the spectrum of academic ability; a desire to teach to those differences makes teaching a difficult challenge. Teachers who continue to try a variety of approaches to assist individuals in learning can make a great deal of progress toward minimizing their students' difficulties in mathematics.

Mathematically Gifted Students

One of the continuing concerns is appropriate education for gifted students. Mathematically gifted students are those who exhibit exceptional mathematical potential in areas such as formulating problems, handling and organizing data, fluency of ideas, originality of interpretation, ability to transfer ideas and ability to generalize. The mathematically gifted are not to be confused with students who are simply good at computational skills. In fact, although many gifted students are also good at computation, many are not. It is also important to be aware that mathematically gifted students may or may not be in local school programs for the gifted.

A pertinent question to consider is: "Is the mathematics curriculum for these students challenging and appropriate?" Administrators and educators need to address this question and begin to provide answers in the form of curriculum and suggestions for teaching mathematically gifted students. Concerns for helping all students work toward their **potential** should guide in the development of mathematical programs for these gifted students. The classroom teacher will be involved in the development and/or implementation of mathematical activities for gifted youngsters.

Mathematically gifted students in the middle school grades are beginning to evidence logical thinking skills which need to be nourished and encouraged. They are asking "Why learn this?" and "What good is it?" The teacher might use applications of mathematics to effectively answer these questions. Concept formation and formalization are of particular importance at this age. Extension of the students' mathematical world is crucial. These mathematically gifted students are in a sense on the brink of life-time preferences. Their teacher can either turn them **on** or **off** to mathematics and may influence their attitudes toward mathematics for the rest of their lives.

Problem Solving

As problem solving becomes the focus of school mathematics in the 80s, the gifted children can be expected to be the leaders in classroom dialogues. They will be the most able in generalizations and applications, in mathematical modeling and insight into problem formulations.

Mathematically gifted students should view mathematics as a relevant part of their world.

> Problems arrive on the suffering student's desk entirely divorced from all context. "Suppose you wanted to multiply 2.31 times 1.7 . . . " Nobody should **want** to multiply these two numbers, but some might **need** to if the calculation modeled an interesting real-life situation (Hilton & Pedersen, 1980).

Applications could be used to motivate and justify mathematics in the classroom. With calculators easily accessible, more real-life situations could be investigated. Along with the use of calculators comes the need for students to learn estimation skills, which could be incorporated into nearly every facet of mathematics instruction.

The newspaper or magazines are great resources for the study of mathematics. Sample problem situations appropriate for mathematically gifted students follow.

Salesperson. Pretend you are a used car salesperson and have sold three compact cars, two middle-size family cars, one luxury car and one pickup truck. You are working on a straight commission and get ________% for each sale. Look in the classified ads section of the newspaper to determine general prices of these kinds of vehicles. Report the prices of the cars, your commission in dollars on each car or truck, the total price of the sales, and your total commission. Which kind of car would you prefer to sell? Which kind of car would you prefer to buy?

The commission can be determined either by the teacher or the students who can call a local car dealer or someone else who has this information. Similar problems might be developed such as one concerning real-estate, *e.g.*, selling of land in particular sections of town, farm-land, apartment buildings, or different sized homes. These and similar activities make use of mathematical concepts of percentages and their practical applications. Gathering data and making decisions on how to organize information are also skills that are built or reinforced.

Stock Market. Stock market problems are always fun for students. Pretend you have $1000 (or $1 million) to invest in stocks. Choose 5 or 6 stocks and determine the number of shares you can buy. Then watch the stock market for two or more weeks. Report on your gains and losses.

Daily access to a newspaper will be necessary. Initially the students will need some help reading the stock market page to know that a price of 132 1/2 is in dollars as is **up 3/8.** How do you find 3/8 of a dollar? Converting fractions to decimals becomes a needed skill. Students must realize they need to find the price of each stock they choose, then determine how many shares of each stock they can afford. They must make decisions on how to organize their information. These decisions entail a good deal of mathematical reasoning and introduce them to real-world problem solving. A look at the gold market, muncipal bonds and the like are also possibilities.

Airline Meals. Problems can be generated from situations relating to planning meals for an airline, train or restaurant for a day. Students, for example, can find out the number of passengers on a 747 and plan meals using food advertising sections of the newspaper. "If I serve milk, how many glasses are in a gallon?" "How can I know how many people want milk?" "If I buy in large quantities, won't I get a discount?" Such questions, as they arise, point out the complexity of the task at hand. Compromises will have to be made and students will be introduced to realistic problems and problem-solving strategies.

Graphs and Data. Discussing and interpreting graphs in magazine or newspaper articles make graphing more meaningful. Examples of statistics and the reporting of data are also easily found. Either the teacher can create problems with the students searching out the specifics or, after some experience, the mathematically gifted students can make up their own problems with solutions to be shared with each other and/or the whole class.

Actual data collection is a good way to get hands-on experience. Simple examples can make use of the characteristics of the members of the class and students can find the mean, mode, median and then graph the data. Characteristics such as height, age in months and shoe size are possibilities. The nature of the middle school grades learner should be considered when choosing such characteristics and selecting the activities to be sure that **individuals** are not made to feel uncomfortable. For example, when collecting data on heights, students may be asked to write down their heights and pass it to the data recorder rather than individually **telling** their heights. Consideration of hair color, eye color, sex and so on will assist students to become aware that there are various **types** of data.

Polls. Construction of a questionnaire for a poll for a class, grade-level or entire school is within the reach of mathematically gifted youngsters. To ensure discussion and more input, the students might work in groups. Together they can decide on topics for questions, wording, open-ended versus forced-choice questions, while the teacher gently guides and points out how decisions can affect the results. Students should be encouraged to discuss how various sampling techniques such as random, stratified and systematic techniques try to minimize bias. The teacher can converse about why television ratings such as A. C. Nielsen's ratings are based on only 1,200 homes in America and how the sample is chosen. A packet of free materials is available from Nielsen's regional office in Atlanta. Have students find articles dealing with various polls. Discussions of inherent bias in telephone polls is relevant, *e.g.*, does every American have a telephone?

When the students are ready to construct their own questionnaires, they should be allowed to choose the topics (within reason) for the questions. The actual questions themselves are not of great importance to the learning experience, but may be to the students. Also students learn from their own mistakes. They will learn that questions that are open-ended, e.g., "What is your favorite ________?", although not a mistake per se, are exceedingly difficult to analyze. Students begin to understand why most polls are composed of multiple-choice or true-false questions.

Weather Prediction. Weather prediction is another topic that may be appealing to mathematically gifted students and can provide meaningful activities. Students might be asked to keep track of weather predictions from television, radio or the newspaper and actual weather every day for one or two weeks. They should discuss the relationship of probability and weather predictions and be able to explain such statements as **30 percent chance of rain.** The students could do a comparative analysis of predictions from the different sources. Which station has the **best** prediction record? The mathematically gifted students could present their study and findings to the whole class since much of the information is pertinent to all students. As a follow-up activity, the teacher might invite a TV meterologist to visit the class or the class might plan a trip to a news room.

Concepts

Most mathematically gifted students can grasp concepts more easily than other students. The teacher should give them both breadth and depth so that they can learn to generalize from more inclusive concepts. For example, the rules for addition and subtraction of signed numbers (integers), are not really new rules, but simply more inclusive ones.

When learning concepts such as area of surfaces, guided discovery teaching may be most appropriate. If the students can derive a formula, they tend to remember it better and, if they do forget, they can usually reinvent it or rederive it. The concepts that make the rules work, not only the rules themselves, are within the grasp of the mathematically gifted students if only someone points them out. The organizational structure of these students' minds and their potentially broader view of the world require that the teacher point out these relationships between the **known** and the **new.**

In classroom discussions, the teacher should look for and encourage the mathematically gifted students to use inductive and deductive thinking skills. Reasoning from the particular to the general (induction), could be encouraged through seeking out patterns, such as in number sequences. Reasoning from the general to the specific (deduction), might be encouraged any time a rule or definition has been established.

Reports and Projects

Mathematically gifted students can become more aware of the breadth of mathematics through reports and projects. The teacher can help these students expand their mathematical horizons. Choices of topics for reports should be offered. Films and filmstrips could be used as well as books. Possibilities of topics include history of particular areas of mathematics—number bases, early numeration systems, computers, the four-color problem, figurate numbers, amicable numbers, perfect numbers, prime and composite numbers, Pythogorean triples, Erastosthenes sieve, Fibonacci numbers, Pascal's triangle and the golden section. Careers in mathematics, biographies of mathematicians, both women and men, are also viable topics. A word of caution is in order here. If students are already doing many reports in other subjects, another vehicle of inquiry may be more appropriate. NCTM's Thirty-first Yearbook (1969) is a good resource for teachers on historical topics. Another resource for ideas in general is the February issue of *The Arithmetic Teacher* (1981) which is devoted to the mathematically gifted student.

Projects such as string art; posters based on topics such as the metric system, numeration systems, graphs of collected data; science-fair type projects; and so on can offer a challenge and a different mode of presentation.

Extra Time

How can the overworked classroom teacher and the mathematically gifted student **fit in** all the extras described above? To begin with, students are probably already **grouped** in some way in the classroom or at least within the teacher's mind. Not every student need to have the same assignments. If students already know a particular concept, why should they be asked to complete an assignment of 40 such problems? The teacher could have them work the last or hardest five or seven exercises to reassure the teacher and the students themselves that they have mastered that concept. Then the students will be able to have the extra time to proceed with the chosen task. Their teacher will also save time by not having to grade 40 or so problems and can in turn devote this time to their other activities.

Reward Excellence

The teacher must try very hard not to make the additional assignments appear to be a **punishment** for attained success. Rather the mathematically gifted students' projects or activities are a **reward** for that success. Sometimes bright students are unintentionally encouraged to be in slower groups so as not to have the burden of more work. Until the students begin to realize that the **extra** is more interesting and in lieu of something, they may balk at the idea of different work. It seems to them that it is merely harder or more trouble than doing 40 of the same type problems. In fact, it is harder but the rewards in understanding are also greater.

Mathematically gifted students can be a real joy. Their teacher should challenge them and encourage them to take risks. At the same time, the teacher must be willing to assume a similar risk often letting students investigate areas of mathematics not totally familiar to him or her. The teacher's efforts initially are great, but by investigating and learning together, the rewards to both the students and their teacher are equally great.

83

Using Support Systems

Rapid sociological and educational changes are placing increasing demands on personnel within a school system. Accompanying these complex changes in our society, and consequently in our schools, are economic considerations that promote the concept of accountability for existing resources. The scarcity of resources and subsequent demands for maximum use of existing resources make it mandatory that school systems establish the priority of the various components of their educational program. School systems should give serious consideration to the desired significance of the mathematics program and allocate funds accordingly. Once these funds have been allocated, every effort must be made to use all of the existing resources to their fullest potential.

The burden of meeting increased expectations from the public when funds are being decreased at the same time is often perceived by classroom teachers to be a burden that ultimately rests on their own shoulders. Although this burden cannot be completely lifted, providing an adequate support system usually lessens the burden significantly.

The potential support system for all classroom teachers is quite large regardless of the size of the school system. It appears, however, that a complex array of factors impedes the effective use of the available support system. The two major factors seem to be (1) knowing where to find assistance and (2) having time to plan for utilizing the assistance once it is located. The principal as the instructional leader of the school can be of great assistance in this area but he or she often has so many other administrative responsibilities that time is not allotted for such support.

This problem is compounded in systems that employ a large number of new teachers. It is difficult to orient these new staff members to all of the existing support available to them. Some school systems use the orientation of new teachers as a vehicle for transmitting information about the available support system.

Another major problem concerns the available time for classroom teachers to use the resources that have been identified. Parent volunteers and student aides make wonderful contributions to the overall program but it is very time consuming for classroom teachers to coordinate their assistance. School systems should continue to give consideration to both of the problems surrounding effective use of support services so that the effectiveness of instruction for each youngster will be enhanced. This section of the guide is written to provide additional assistance to teachers in locating and using various support systems that are available.

The support services available to teachers can be classified as human resources and material resources. These two types of resources are found at local, state and national levels. A summary chart is provided for easy references (see Chart A).

Human Resources

Probably the most significant of all the human resources available to teachers and students are the professional staff members of the individual schools. The teacher must often take the initiative so that these individuals will be aware of the teacher's need in a particular area. One technique that is used by some schools is to assign new teachers to veteran faculty members who will assist them in utilizing the various aspects of the existing support system. Counselors, media center specialists and special education specialists provide additional support for the classroom teacher.

One of the major areas where the teacher might feel a need for additional support is the education of students with special needs. Questions about the classroom teacher's role in planning for the provision of services for students with special needs and available assistance from these individuals can be answered by the special educator or principal. For example, teachers of the gifted are sometimes available to discuss the special needs of the most able students. They may be able to give the classroom teacher ideas or materials to use to help these students reach their potential. See sections on the gifted and on students having special difficulties in mathematics.

Professional support is also found at the system level in the form of consultants in middle grades education and mathematics. They may also be available in other areas such as art, music and special education. The Cooperative Educational Services Agencies (CESA) can also provide many of these services. Other very important sources of professional support are various professional organizations. These organizations, whether general or directed toward specific interests, such as the Georgia Council of Teachers of Mathematics, are systems of support consisting of people who have similar interests and goals. These organizations often sponsor inservice programs for teachers.

Other human resources on the local level include business leaders, service clubs, and city and county officials. Often overlooked, both students and parents are also sources of support for the classroom teacher. They can both be helpful in making materials, learning centers, math games, bulletin boards, *etc.* An additional benefit is that children of parents who help prepare materials are eager to use these materials in the classroom. Often parents need help knowing **how to help**, but the effort on the part of the teacher is worth it. These trained parents are valuable resources and are motivation factors for the students.

Parents are often willing to take time to come to speak to the class about their jobs and how they use mathematics daily. Computer people, engineers and architects are obviously heavy users of mathematics, but there are many other occupations to consider. A farmer, for instance, uses mathematics every day in selecting types and amounts of fertilizers, determining production and cost per acre and totaling profits. A panel of parents might be asked to discuss such a topic.

Parents can also reinforce concepts learned in the classroom. They want to be a part of the education of their children. Sample Parent Cards in the activities section of this guide may help the teacher and the parents begin. Parental interest should have an impact on the students' learning. The teacher should also consider legislators on the state and national levels as sources of support.

Material Resources

Although human resources can greatly enhance the mathematics program of a school, there is also a need for an adequate support system in the area of material resources. Materials other than the textbook are necessary for middle grades students who have grown up in a multimedia, technological society.

The most available material resources for the classroom teacher are those materials in the school media center. Professional books and materials for teachers as well as students are available. Audiovisual equipment (such as overhead, movie, opaque and filmstrip projectors; record players; or television sets) are usually located in the school—often in the media center. If these materials are not available in the school, they may be obtained from the school system. Videotape equipment, computers and other technological developments are available in many school systems, homes, local business firms, Cooperative Educational Services Agencies and nearby colleges and universities. Both teacher and student use of new technological resources can greatly enhance students' attention, motivation and learning processes.

Somewhere in each school, usually in the media center, are commercial catalogues of educational material. Sometimes materials are ordered for the individual teacher and sometimes for the school. Often there are closets where mathematics materials, such as compasses, rulers, meter sticks, attribute blocks, clocks, protractors, models of polygons and polyhedra, are stored. The teacher may have to consult with the principal, media center specialists or other teachers to locate these materials.

Some libraries have professional journals such as *The Arithmetic Teacher* or *The Mathematics Teacher* (NCTM publications). These journals have many timely articles designed to assist both the new teacher and veteran alike.

Another aid in your media center is the Resource File for the community. This file will help the teacher locate various types of community support such as guest speakers from the business world who are willing to come to classes or services of clubs or other groups.

Curriculum guides such as this one can often be found in the school media center. Such guides can help the teacher determine the system's expectation for mathematics instruction and guide the teacher in planning for instruction. A more detailed description of material resources available can be found in the Appendix entitled *Instructional Resources to Support the Middle School Mathematics Program.*

Community resources, both human and material, are probably the most unused of all the aspects of a support system for teachers. Many agencies have fantastic learning materials that are prepared for school use. Banks, utility companies and governmental agencies such as the Soil Conservation Office are examples of agencies that may have these materials. These materials not only provide additional resources for teachers but also assist students in translating academic learning into solutions of practical problems found in daily life.

One obvious problem in obtaining many of these resources is money. There are often funds that are available if requested. Budget questions, both short and long term, can often be answered by the principal. Money for special materials of classroom teachers can come from local school budget, local PTA budgets and system level budgets. There is often system money earmarked for mathematics and science materials. In addition there are often state and federal monies set aside for mathematics instruction. Money or materials may be available through special education budgets for students who have special needs, such as braile material or tapes for the blind, or instructional modules for students with specific needs. Community groups may be helpful in finding money for materials or special equipment. The Lions Club, for instance, may obtain eye glasses for students who otherwise could not afford them.

In summary, every school system in Georgia and every classroom teacher need and can use additional resources to bring about increased student achievement. These resources, both human and material, exist in abundance at local, state, and national levels. The extent to which school systems and individual teachers can work around obstacles in locating these resources and finding time to plan for their use will determine the significance of support systems in the educational opportunities of students in a given school district or classroom.

Chart A

	Local	State	National
Human Resources			
Professional Educators	Within School Fellow Teachers Lead Teachers Principals Media Center Specialist Teacher Aides Guidance Personnel (Counselors) Special Education Teachers/Specialists School System Specialists such as Art, Music, Special Education Consultants or Coordinators University and College Personnel Cooperative Educational Services Agency (CESA) Professional Associations	Consultants in Mathematics Consultants in Middle Grades Other Area Consultants such as Special Education, Art Georgia Council of Teachers of Mathematics (GCTM) General Education Associations such as Association of Classroom Teachers	Federal Education Agencies National Council of Teachers of Mathematics (NCTM) National Education Association
Noneducators	Students Parents Business Leaders Service Clubs and Other Civic Groups Board of Education City and County Officials Other Individuals Listed in Community Resource File in School	Legislators Georgia Board of Education	Legislators
Material Resources	Local Curriculum Guides Libraries County or City Art Resources Media Center Film Libraries Community Resource File Found in School	Georgia Department of Education Publications *Essential Skills for Georgia Schools* *Mathematics for Georgia Middle Grades* GCTM Publications Film Libraries	Various Educational Journals NCTM publications *Arithmetic Teacher* *Mathematics Teacher*

Sample Activities

Activity Ideas

The first three sets of activities are in the form of a compilation of ideas for activities arranged by objective. Activities are not included for every objective. The strands for which there are activities are

Set 1 — Sets, Numbers and Numeration (S/N/N)

Set 2 — Operations, Their Properties and Number Theory (O/P/N)

Set 3 — Geometry (G)

The obvious disadvantage of listing ideas for activities by objectives is that most activities will assist students in meeting several objectives. The ideas are listed under those objectives for which they seem most appropriate.

The fourth set of activities is entitled, "Personal Characteristics" and is a cross-strand activity although it has an emphasis on probability and statistics. Another feature of this activity is that it may be used in its entirety or portions of the activity may be used in another unit such as graphing.

Sets, Numbers and Numeration

S/N/N 1
Use the language of sets to describe and organize information

1. Have the pupils name some sets of objects in the room such as the set of desks, set of pupils, etc.
2. Have the pupils name sets of groups of which they are a part.

   Examples

   Set of pupils who attend _________ school.

   Set of pupils who ride the bus.

   Set of pupils who bring their lunch.

   Let each pupil read aloud one of his categories and have those who are members stand. Also let the ones who are not members of each set stand to represent the complement of the given set.

   If a pupil does not name a set that is not well defined, the teacher may describe a set such as "the set of pupils who are wearing new dresses" or "the set of pupils who talk too loudly in the lunchroom." The pupils will easily see that these sets are not well defined because "new" and "loudly" are not specific.

3. To teach the descriptive method of designating a set, the pupils could be asked to describe sets such as the following.

   $A = \{10, 100, 1000, \ldots\}$

   $B = \{10, 20, 30, \ldots\}$

   $C = \{a, e, i, o, u\}$

   When they describe the sets as

   A is the set of powers of 10,

   B is the set of multiples of 10 and

   C is the set of vowels,

   they will not only have had practice in using the descriptive method but also in observing the common property of given sets.

4. Give the pupil opportunities to find subsets of an infinite set of numbers. For example, let A be the set of whole numbers.

   Have the pupils list certain subsets such as the following.

   B = the set of even numbers

   C = the set of multiples of 5

   D = the set of factors of 12

   E = the set of numbers < 3

   F = the set of numbers > 6

   G = the set of numbers > 3 and ≤ 6

   H = the set of numbers < 3 and ≥ 6

   I = the set of numbers such that $2N = 6$

   J = the set of numbers such that $N + 6 < 10$

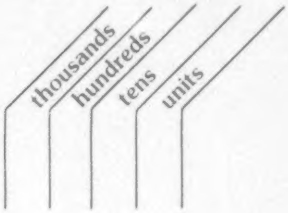
S/N/N 4
Use expanded exponential form to demonstrate place value of the decimal system

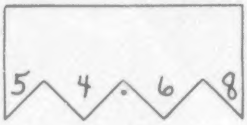
1. Review the place value code for writing numerals and extend the idea that place value names are a convenient way of identifying the number of subsets that can be made by grouping objects into subsets of ten. Stress that the convention for writing numerals requires that the count of tens be recorded to the left of the count of the units. The relationship between places is 10 times the one to the right. The place value chart is a good device to illustrate this convention and can serve as readiness for developing exponential notation.

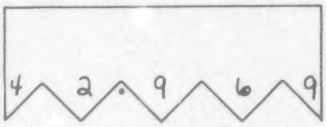
   The teacher will need to provide experiences with physical objects to build concepts of place value for those pupils who have not acquired them by the end of the primary grades.

2. One activity that might be used with a group of students or the entire class would use teacher-made cards that could be held against the chalkboard and be seen from any spot in the room. A five by seven index card could be used. A large decimal point is placed on the chalkboard and the cards moved so that the decimal point appears in a different place in the number. Students can then read the number aloud or write it in expanded form, etc.

   Examples of cards are

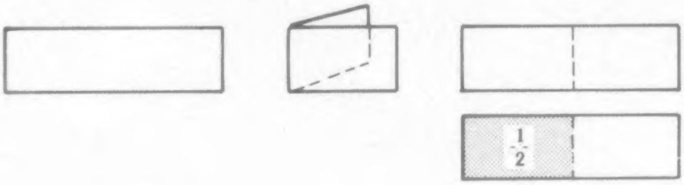




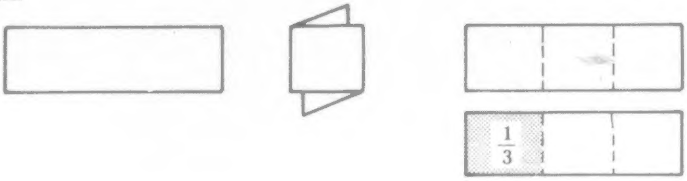


S/N/N 5
Name the pair of numbers associated with fractional parts of (a) units, (b) sets

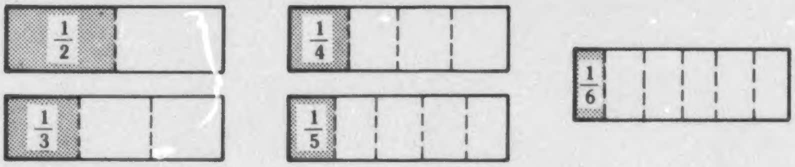
1. Prepare fraction makers for pupils by cutting out strips of paper which measure about 3 inches by 8 inches. (a) Give each pupil a strip and direct him or her to fold along a line so that the ends of the strip fit together. Then have the pupil unfold the strip, shade one subregion and write the fraction name for the pair of whole numbers which describes the experiment.



   (b) Give each pupil a second strip and direct each to fold the strip as in the illustration below. This activity will involve some trial and error. The teacher may demonstrate with a strip; however, let each student work with folding until each has accomplished the fitting together of the parts of the strip. Have the pupil unfold the strip, shade one subregion and write the fraction name which describes the experiment.



(c) Continue having pupils fold strips and shade one of the resulting subregions until on each pupil's desk there is a set of strips, shaded and labeled, as pictured here.



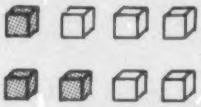
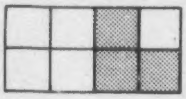
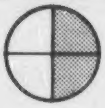
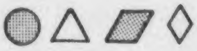
Have students discuss their observations as they fold the strips.

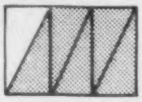
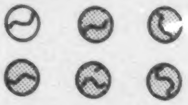
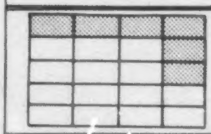
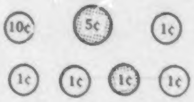
Out of such discussions can also arise statements such as, "If I shaded all of the pieces in each strip then I would have 2/2, 3/3, 4/4, 5/5, and 6/6." Also, pupils may notice that two 1/5s and three 1/5s make five 1/5s, and the like. The teacher's continued encouragement to pupils to talk about what they see can provide the first intuitive recognition of relationships among fractions. The encouragement to use number names when talking about pieces will help to develop the language of fractions. However, one must be careful not to insist on mastery of facts about fractions until pupils possess the concept of fraction.

2. Repeat activity 1 except let the strips be of different length or width than the strip which was provided for the previous activity. Although one does not explicitly point out to young learners that fraction numerals such as 2/3 represent an abstract idea, the teacher needs to provide experiences with different models of units and with quantifying parts of those units. Thus, the pupil abstracts the idea of fraction without regard to any particular model.

S/N/N 6
Name the pair of numbers associated with fractional parts of (a) units, (b) sets

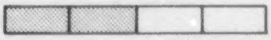
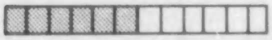
Have students partition units and sets so that they understand the concept of fractional parts in each context.

Partition of Unit (Set)	Number of Pieces (or members) in Unit Set	Number of Pieces (or members) Shaded	Fraction Name
	8	3	$\frac{3}{8}$
	8	3	$\frac{3}{8}$
	4	2	$\frac{2}{4}$
	4	2	$\frac{2}{4}$

Partition of Unit (Set)	Number of Pieces (or members) in Unit Set	Number of Pieces (or members) Shaded	Fraction Name
	6	5	$\frac{5}{6}$
	6	5	$\frac{5}{6}$
	20	6	$\frac{6}{20}$
	20	6	$\frac{6}{20}$

S/N/N 7
Identify the set of equivalent fractions associated with a given point on a number line

1. Working with paper strips as fraction makers (See Objective 5.), have pupils partition the units by folding or cutting them; then have them shade some of the pieces and identify the associated number pairs and fraction symbols as indicated in the chart.

Partition of Unit	Number of Pieces in Unit	Number of Pieces Shaded	Fraction
	2	1	$\frac{1}{2}$
	4	2	$\frac{2}{4}$
	6	3	$\frac{3}{6}$
	8	4	$\frac{4}{8}$
	10	5	$\frac{5}{10}$
	12	6	$\frac{6}{12}$

Ask "What can you say about the shaded parts of the units (strips)?"

Record pupil responses on the board and discuss. Among responses may be one such as "The same amount of paper is shaded in each strip." If asked, "Is the fraction $\frac{1}{2}$ the same as the fraction $\frac{2}{4}$?", the response should be "No, because the partitions are not the same." Further discussion should lead to the agreement that although the units are not the same they are equivalent (same amount of paper) and although the partitions are not the same, the shaded portions of each unit are equivalent (same amount of paper). Thus, the number pairs or fractions are said to be equivalent, and one can say of the following sets that the fractions are equivalent to each other.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\right\}$$

Ask, "Are there other number pairs or fractions which we could include in this set?" If necessary, continue partitioning units and shading parts until the pupils decide that there is no **last** fraction in a set of equivalent fractions. Thus, one writes the following.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \ldots\right\}$$

2. Expand or build on activities above to provide practice in generating members of a set of equivalent fractions. Set up work sheets as follows.

   In each of the following exercises, write five other fractions that belong to the same set as the given fraction.

   (a) $\left\{\frac{1}{2}, \qquad\qquad\qquad\qquad \right\}$
   (b) $\left\{\frac{3}{4}, \qquad\qquad\qquad\qquad \right\}$
   (c) $\left\{\frac{5}{3}, \qquad\qquad\qquad\qquad \right\}$
   (d) $\left\{\frac{5}{8}, \qquad\qquad\qquad\qquad \right\}$
   (e) $\left\{\frac{8}{6}, \qquad\qquad\qquad\qquad \right\}$
   (f) $\left\{\frac{5}{5}, \qquad\qquad\qquad\qquad \right\}$

S/N/N 10
Order any two or more given rational numbers (whole numbers, decimals, fractions and mixed numbers)

1. A variation of a commercial game is a versatile game called Ranko. It consists of a deck of cards which may be whole numbers, common fractions, decimals, integers, numbers written in scientific notation or any other topic in the curriculum where ordering is involved. The ranko board can be made from wooden strips with five diagonally cut slots to hold the cards or from upside down egg cartons that have been slit across the top of each egg holder.

   Once the students have learned the game, they can concentrate on the ordering.

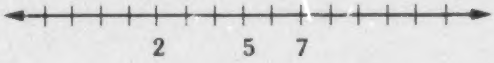
   Use groups of 3 to 5. Give each student a ranko board. Deal five cards to each student and have them place the cards in their ranko board in the order in which they are dealt. The students then take turns drawing one card. They can replace any of their five cards with this one—the object being to move toward putting their cards in order.

   An unusually slow group could simply see who could order theirs without drawing. If several students are playing, the first one to order their cards would win. It does not, however, have to be competitive.

S/N/N 17

Order any two or more given integers

1. If a pupil has had many experiences with the number line he or she could be asked to consider a number line that looks like this.



   Ask him to fill in other numbers for the indicated points on the line so that he has a chance to suggest 1, 2 and 3. The number line provides an excellent opportunity for developing the concept of opposites. Given a number line with unit segments to the right corresponding to steps to the right, and unit segments to the left corresponding to steps to the left, the teacher should ask questions such as the following.

   (a) What is the opposite of 2 steps to the right? The opposite of $^{+}2$ is _________.

   (b) What is the opposite of 5 steps to the left? Then the opposite of $^{-}5$ is _________.

   Tell the pupil that the union of these two sets is the set of integers $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2\}$

   Pupils themselves can suggest many activities involving the recording of gain and loss, such as the number of yards rushing in football games, net profit or net loss in business and similar examples.

2. Have the pupils locate places on the map that are above and below sea level. They can describe the locations using integers.

3. An activity that will appeal to older pupils is the use of a model from science in which the drawings represent an empty field, a bucket containing positive particles and a bucket containing negative particles. As the pupils place positive and negative charges into the field they will observe the results of combining particles of opposite charges. This gives the teacher an opportunity to emphasize the word **opposite** as it relates to the integers. The concept of neutralization adds much to this activity. The pupil may draw a circle around the neutralized particles (±) and this gives insight into combining or adding integers. See activities from the strand, "Operations, their Properties and Number Theory."

4. A game could be played with all pupils standing on a stair landing. Each pupil draws a slip of paper telling him to go up 5 steps, down 9 steps, up 4 steps, down 4 steps and the like. In this activity the teacher should ask questions to develop the idea of opposites. The pupil should see that since the $^{+}4$ (up 4 steps) followed by $^{-}4$ (down 4 steps) leaves him in his original position, that $^{+}4$ and $^{-}4$ are opposites, that is, that $^{-}4$ is the opposite of $^{+}4$ and $^{+}4$ is the opposite of $^{-}4$.

5. Have the pupils record the results of placing a thermometer in liquids which will cause the thermometer to fluctuate. Be sure that it is placed in a liquid which will cause the temperature reading to drop to below 0°C. Such a liquid would be a solution of water and alcohol which has been refrigerated and cooled below 0°C. Encourage the pupils to discuss the measures of temperatures lower than 0°C.

6. Another activity enabling pupils to have experiences which involve positive and negative integers is the postman game. The game involves pupils being homeowners and one pupil acting as postman. The postman delivers checks and bills. Each homeowner keeps track of income and outgo by treating the amounts on checks as positive numbers and amounts on bills as negative numbers. The game can be expanded into the study of operations on integers as the postman becomes confused and delivers the wrong mail to several people.

Operations, Their Properties and Number Theory

O/P/N 3
Generalize results of operations with odd and even numbers

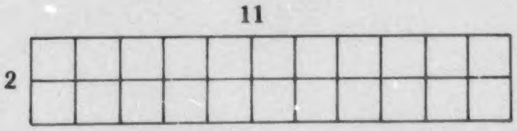
1. Prepare a worksheet or transparency using the following suggestions.
   a. Circle each of the even numbers in set A.
      $A = \{6, 9, 10, 4, 11, 12, 15, 16, 17, 20\}$
   b. Express each of the even numbers in the set as the product of 2 and some counting number. For example, 6 = 2 × 3.

   Pupils should work together to solve problems using concrete materials. They should be encouraged to discuss the question "Do you think every even whole number can be expressed as 2 times some whole number?"

2. Develop a class discussion around the following suggestions.
   a. Circle all of the odd numbers in set B.
      $B = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, 15\}$
   b. Express each odd number in set B as the sum of an even number and 1. For example, 3 = 2 + 1.
   c. Can you find an odd number which is not one more than some even number?

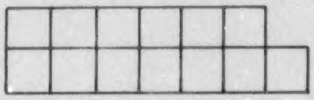
3. Present questions such as the following to pupils to encourage them to generalize the results of operations with even numbers and odd numbers.
   a. Is the sum of any two even numbers even or odd?
   b. Is the sum of any two odd numbers even or odd?
   c. Is the sum of an odd number and an even number odd or even?
   d. Is the product of two even numbers even or odd?
   e. Is the product of two odd numbers even or odd?
   f. Is the product of an odd number and an even number even or odd?

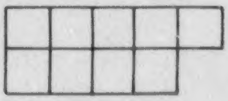
4. a. Cut strips of lightweight cardboard. (Old manila folders are good weight.) Make strips 2 by n, where n is any natural number.



This strip represents the even number 22, 2 × 11. By folding accordion fashion it can be used to represent any even number, 2 × n. Another such strip may be used to represent any other even number, 2 × m, where m is any natural number. By joining the two strips end to end, pupils can demonstrate that the sum of any two even numbers is an even number.

b. Make representations of two odd numbers from strips of paper as described above.


(2 × 6) + 1 = 13


(2 × 4) + 1 = 9

Let pupils use these models to show that 13 + 9 represents an even number. By folding accordion fashion these models can be used to represent any two odd numbers less than 15 and 11, respectively, and can serve to help pupils realize that the sum of any two odd numbers is an even number.

O/P/N 4.
Identify prime and composite numbers

1. Use a transparency or ditto copies of a hundreds square, a 10 × 10 grid. Begin with the number 1 and numbering across to the right and back again from left to right label the squares 1-100.
2. The Sieve of Eratosthenes is explained in most textbooks as a technique for finding prime numbers less than a given whole number, usually 100. Hence a detailed account is not needed here. The following steps summarize the method.
   a. Cross out 1 because it is not prime by definition.
   b. Cross out all multiples of 2 except 2.
   c. The next three prime numbers in order are 3, 5, 7. Cross out all multiples of 3, 5, 7, except 3, 5, 7.
   d. Circle the remaining numbers. They are prime numbers less than 100.

   It is not necessary to instruct pupils to cross out multiples of 4 or 6 because they were crossed out as multiples of 2 and of 3.

   Discuss why you need not continue the steps above to include crossing out all multiples of 11, the next prime number. (Any number less than 100 which has a factor of 11 has another factor less than 11 and hence has already been crossed out.)

   Make a chart of all prime numbers less than 100, and display it in the room.
3. Develop a class discussion around the following questions.
   a. Circle the prime numbers in each set.
      $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
      $B = \{2, 4, 6, 8, 10, 12\}$
      $C = \{1, 3, 5, 7, 9, 11, 13, 15\}$
   b. Can you find a prime number that is an even number?
      Can you find an odd number which is not prime?
      Can you find two consecutive numbers that are prime?
      Can you find three consecutive odd numbers that are prime?
   c. Tell whether each of the following statements is true or false.
      1. All even numbers are composite.
      2. All odd numbers are prime numbers.
      3. One is a prime number.
   d. How many pairs of twin primes can you find? (Twin primes are primes whose difference is 2.) For example, 5−3 = 2. Hence 3 and 5 are twin primes.
4. Ask questions such as: Can you write names for all even numbers 14 or greater as sums of two prime numbers in more than one way?

   Example:

   14 = 7 + 7 = 3 + 11
   16 = 3 + 13 = 11 + 5
   18 = 5 + 13 = 7 + 11
   20 = ____ + ____ = ____ + ____
   etc.

O/P/N 5
Determine the factors of any whole number

1. Have pupils find the sum of the divisors of 6, other than 6 itself. (1 + 2 + 3 = 6.) Find another counting number that is equal to the sum of the divisors that are less than the number itself. The Greeks called such numbers **perfect numbers**.

   A perfect number is one that equals the sum of its proper divisors, and the proper divisors of a number are all those except the number itself.

   The first four perfect numbers are 6, 28, 496, 8128. Note that the first perfect number is a single digit numeral, the second is a two digit, the third is a three digit, and the fourth is a four digit — all in base ten numeration. However, the fifth perfect number has eight digits, 33550336.

   Ask groups of pupils to confirm the fact that these five numbers are perfect numbers.

   Arrange the numerals vertically and study the digits. What pattern do you observe in the last digits of the numerals? Only 23 perfect numbers have been found. The largest of these numbers requires 6,800 decimal digits to write the numeral.

   Other interesting ideas concerning the perfect number may be found in books listed in the reference section of this guide.

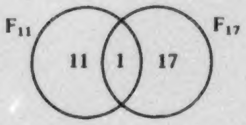
2. Ask at least five pupils to factor the same number. Choose numbers appropriate for the pupils' abilities from the set given.

   $$\{54, 75, 120, 168, 225, 363, 432, 576\}$$

   Compare the five factorizations of each number. Were the factorizations all the same? If not, how were they different? Discuss the fact that the order of factors makes no difference in the product, hence the pupils should suspect that every composite whole number can be expressed as a product of primes in just one way (except for order.) This fact is called "The Fundamental Theorem of Arithmetic."

O/P/N 6
Determine the greatest common factor of a set of numbers

1. a. Ask the pupils to find the greatest common factor of 11 and 17.



Since $F_{11} \cap F_{17} = 1$, 1 is the greatest common factor, in fact, the only common factor.

If GCF (a,b) = 1, a and b are whole numbers, then a and b are relatively prime.

Circle the pairs of numbers that are relatively prime and tell why they are or are not relatively prime.

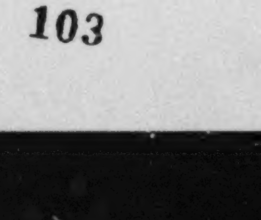
(3, 5), (2, 8), (3, 27), (49, 12), (3, 8), (7, 17)

b. Ask the pupils to draw Venn diagrams to show the intersection of the sets.

$A = \{1, 3, 5, 7\}$

$B = \{2, 3, 4, 5\}$

Solution

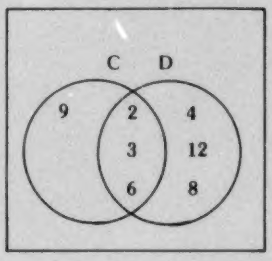


c. Ask the pupils to draw the Venn diagrams to show the intersection of the sets.

C is the set of factors of 18

D is the set of factors of 24

Solution



Name the greatest common factors of 18 and 24.

O/P/N 7

Determine the least common multiple for a set of numbers

1. Ask pupils to do the following.

a. Write the set of natural number multiples of 4, of 6, of 8 that are less than 100.

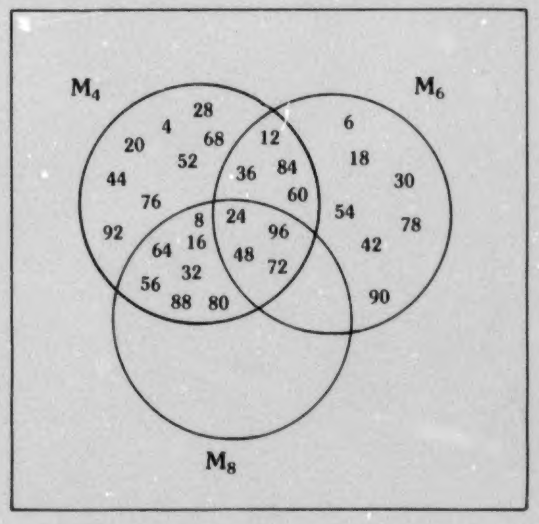
$M_4 = \{4, 8, 12, \_\_\_\_, \_\_\_\_, \text{etc.}\}$

$M_6 = \{6, 12, \_\_\_\_, \_\_\_\_, \_\_\_\_, \text{etc.}\}$

$M_8 = \{8, 16, 24, \_\_\_\_, \_\_\_\_, \_\_\_\_, \text{etc.}\}$

b. Show the common multiples of M_4, M_6 and M_8 using Venn diagrams.

Solution



c. Find the least common multiple of 4, 6 and 8 that is LCM (4,6,8) = ______.

d. List the members of each of the following.

a. $M_4 \cap M_6$

b. $M_6 \cap M_8$

c. $M_4 \cap M_6 \quad M_8$

O/P/N 8
Identify and continue number patterns

1. As a device for further illustrations of operations with whole numbers, the teacher may make mimeograph copies of the calendar such as the following.

A 7s Calendar

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Ask pupils to study the arrangement of the numbers looking for patterns. Ask in particular for the pattern for the numbers in the first column; the pattern for column four; the pattern for a diagonal.

Twenty-eight minus what number equals 22? Twenty-two minus what number equals 16?
16 − ? = 10. 10 − ? = 4. Ask pupils to discover other patterns. Also make other calendars.

2. Give students tasks like the following so they can discover patterns that exist in computational problems; calculators are very appropriate to use in this activity since the objective concerns the patterns and not the computational skills.

Continue squaring the numbers given below until you can discover the pattern, then predict the answer to the next problem. Check to see if your guess is correct. Do you see why this works?

$1^2 = 1$

$11^2 = 121$

$111^2 =$

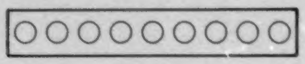
$1111^2 =$

$11111^2 =$

Complete these multiplications to find some interesting patterns.

7 × 7 =	4 × 4 =
67 × 67 =	34 × 34 =
667 × 667 =	334 × 334 =
6667 × 6667 =	3334 × 3334 =
	33334 × 33334 =

3. There are many number pattern activities of interest to students. The commercial names for two of the standard ones are the Tower of Hanoi and the Eight Man Puzzle. They can be made by using materials easily available to students and teachers. For example, the Eight Man Puzzle can be made by placing holes in a strip of board and using golf tees for the pieces.



Students will generally find a vertical pattern in the minimum number of moves needed before they find the horizontal pattern that can be abstracted.

$$\begin{array}{cc} 3 & \\ & 5 \\ 8 & \\ & 7 \\ 15 & \\ & 9 \\ 24 & \end{array}$$

1	$1 \times (1+2) = 3$
2	$2 \times (2+2) = 8$
3	$3 \times (3+2) = 15$
4	$4 \times (4+2) = 24$
n	$n \times (n+2) =$

These activities and patterns are also valuable in helping students to use variables to express the pattern abstractly $n(n+2)$.

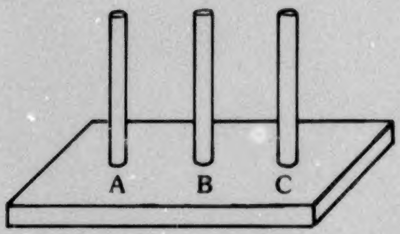
Directions

Place a set of pegs on each side of the board leaving the center hole open. Each set consists of 4 pegs of the same color. The object of the game is to transfer each set of pegs to the opposite side—thus reversing the original setup. This must be done by either moving a peg one hole forward or jumping a peg forward over another peg. Moving or jumping backward is not permitted.

Students can be encouraged to develop a system for analyzing this problem. For example, they may be encouraged to use one peg on each side which takes three moves. They could then use two pegs on each side which takes eight moves. The technique for completing the task is more readily seen when a smaller number of pegs is used. The completed chart would look like the following.

Number of pieces on each side	**Minimum number of moves**
1	3
2	8
3	15
4	24
.	.
.	.
.	.
8	80
n	$n(n+2)$

4. A similar activity with a more difficult pattern is the Tower of Hanoi. This game is played on a board containing three pegs or nails.



There are a number of discs of varying colors and sizes that fit over the pegs or nails. They are to be moved from one station to another without placing a larger disc on a smaller one and they may only be moved one at the time. The rules are as follows.

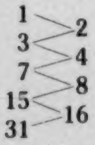
1. Stack the five discs in the form of a pyramid (put the largest on the bottom and the smallest on top) in space A.

2. You may move only one disc at a time. It must be the top disc on any stack.
3. You may move a disc from one space to any other space, but never place a larger disc on top of a smaller one.
4. When you have finished moving the stack of discs to another space (say C) they should again be stacked as a pyramid.

The pattern that appears as students record their data for the minimum number of moves needed is

0	0
1	1
2	3
3	7
4	15
5	31

2 (3) + 1 = 7
2 (7) + 1 = 15
2 (15) + 1 = 31



2 $2^2 - 1 = 3$
3 $2^3 - 1 = 7$
4 $2^4 - 1 = 15$

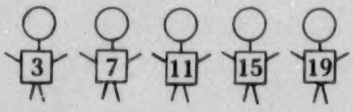
Again, a vertical pattern is usually found first. The vertical pattern in this one is $2n + 1$. The horizontal pattern is $2^n - 1$.

5. Another good activity that might be done to generate a pattern of numbers for students to analyze is an activity using the diagonals of a polygon.

Each student is given a sheet of polygons and asked to guess the total number of diagonals in each one. Following the opportunity to guess, the students will draw in the diagonals from one vertex, count as they draw, and record the number in the following chart. The students are then asked to predict for a polygon of n sides $(n-3)$. Following this, the students will follow the same procedure for the total number of diagonals $\frac{n(n-3)}{2}$.

Number of Sides	**Number of Angles**	**Number of Diagonals From one Vertex**	**Total Number of Diagonals**
4	4	1	2
5	5	2	5
6	6	3	9
7	7	4	14
.	.	.	.
.	.	.	.
10	10	7	35
n	n	$n-3$	$\frac{n(n-3)}{2}$

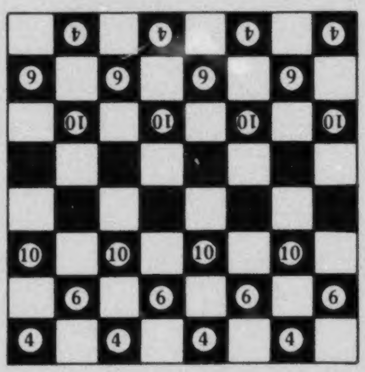
6. Cards with "people" showing arithmetic sequences can be used to give students practice in finding rules and solving sequences.



O/P/N 9

Demonstrate immediate verbal recall of any basic facts

1. A game of checkers may be used to reinforce number combinations. Regular rules for checkers may be used; however, the red and black discs are numbered as shown on the diagram. Pupils score this game by adding the numerals on the checkers as jumps are made. For example, if a red 4 jumps a black 6, the red scores 10. A double jump would cause the pupil to add three numbers. When a checker reaches the King row, it doubles in value. The player with the highest total score wins. Different numerals may be used according to the drill that is needed by pupils. This game may be adapted to multiplication, also.



2. Use magic squares to give students practice in recall of basic facts. The teacher may illustrate the magic square using the illustration below.

Magic square using numbers 1 through 9

4	3	8
9	5	1
2	7	6

Find the sum of each row of numbers; of each column of numbers; of each diagonal line of numbers. Ask if the sum was the same in every case. Make duplicate copies of squares. Ask pupils to try rearranging the numbers and add again. What happens?

Use a magic square using numbers 2 through 10, such as the following.

9	4	5
2	6	10
7	8	3

What is the sum of each row? Each column? Each diagonal? Add in opposite directions. What happens?

Illustrate by using the overhead projector or draw on the chalkboard a 4 by 4 magic square using numbers 1 through 16 as follows.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Ask pupils to find the sum of the numbers in row two, the sum of the numbers in column three and the sum of the numbers on the diagonal beginning with 16. Reverse the order of the addition. What happens?

Ask pupils to draw other 4 by 4 squares using numbers 1 through 16. Is the figure thus formed a magic square?

O/P/N 11

Compute efficiently—both with and without a calculator—using whole numbers, fractions, decimals and negative numbers

There are many games that can be used to give students practice on their computational skills. Although they cannot take the place of instruction in these skills, they can be used to reinforce and are a pleasant change for students who are reluctant to practice these skills when given pages and pages of drill sheets.

1. One game uses a game board and three dice.

   The game board is numbered from 0 to 30. The students roll 3 dice and use the numbers to produce one of the numbers on the board. The student with the most tokens on the board at the end of the game wins.

   Example

   $(5 \times 3) + 1 = 16$

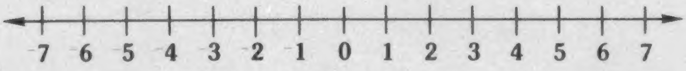
   $(5 + 1) \times 3 = 18$

   $(5 - 3) - 1 = 1$

2. Many of the games are good because the teacher can individualize the game by the set of cards given to the various groups of students. An example of this would be the variations of bingo, concentration, rummy, etc. The level of difficulty is controlled in the makeup of the board or cards.

   A background for understanding operations involving negative numbers can be established by using the following activities as models.

   If one travels a number of miles from one point to another, he or she may travel part way east and part way west. Using arrows, → means east and ← means west. The arrows may be placed over the numerals.



Example

Mr. Jones travels east 4 miles and from that point travels west 6 miles. Where is he with respect to his starting point? Two miles west of where he started. The point here is not how far he has traveled, which obviously is 10 miles, but where he is with respect to his starting position.

Many of the same type problems may be used until the pupils discover the rule, such as the following.

(a) If he travels 6 miles east, then 4 miles west, what is the distance to the starting point?

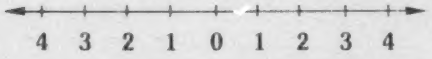
(b) If he travels 4 miles west, then 5 miles east, what is the distance to the starting point?

When the teacher thinks the pupils have the idea, proceed with the recording. Since we name the associated operation addition in this case, the recording for the first illustration given is 4 + 6 = 2. Read this as "Positive 4 plus negative 6 equals negative 2."

After having learned the addition processes of integers, the pupils may be introduced to the subtraction process as the inverse of addition. The teacher may continue using the east - west idea on the number line as developed in the addition process.

Example

For the number sentence 2 − 3 =



Using addition, the inverse of subtraction, one would say, "What has to be added to positive three to get to positive 2?" One would obviously have to travel west one place so the missing addend is negative 1.

Enough of these activities may allow the pupils to get a clear understanding of the idea.

Example

$^{-}3 - 1 =$

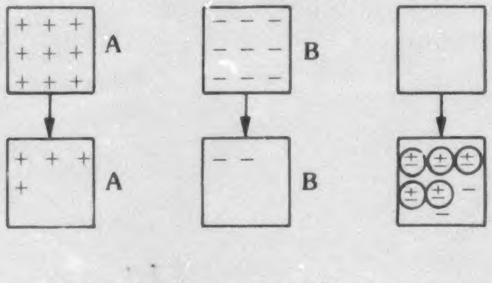
The question is—"What must be added to 1 to get to 3?" Since the move is two places to the west, the missing addend is 2.

Continue with many similar examples.

Probably at the level when directed numbers are introduced, some pupils will have had experience with positive charges attracting negative charges and neutralizing each other. The teacher may wish to show neutralization with charged particles using a science demonstration.

The teacher could also demonstrate the idea by using an illustration of two buckets, one bucket A, containing an indefinite number of positive and the other bucket, B, an indefinite number of negative particles. The question may be asked, what will be the result if seven negative particles from bucket B are placed in an empty bucket, C, and then five positive charges from bucket A are added? Each negative particle will attract a positive particle, and they will neutralize.

This neutralization can be shown by drawing a circle around each pair.



When 5 positive particles were placed in bucket C and 7 negative particles were also placed in bucket C, 5 positive and 5 negatives were attracted to each other and 2 negative charges are left.

The pupils may do many other activities such as the following and in each case ask, "What is the end result?"

ADD 4 positive charges to a bucket containing 8 negative charges

ADD 5 positive charges to a bucket containing 5 negative charges

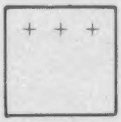
ADD 8 negative charges to a bucket containing 3 positive charges

When the teacher is sure that the pupils understand the idea, then proceed to the recording of the data in number sentences.

For example, the recording of the illustration given at the beginning of this activity is ${}^-7 + {}^+5 = {}^-2$.

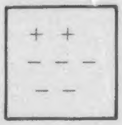
The idea of charged particles as in the addition of integers may be applied in a model for subtraction.

A bucket contains 3 positive charges.



What would have to be added to it to make the bucket have a charge of negative 2?

Obviously 5 negative charges are needed as it takes 3 negatives to neutralize the 3 positives and 2 more negatives to make the bucket have a negative charge of 2.



The addition sentence then would be ${}^+3 + \square = {}^-2$. The replacement for $\square$ was found to be ${}^-5$; therefore, ${}^+3 + {}^-5 = {}^-2$.

The related subtraction number sentence would be

${}^-2 - {}^+3 = {}^-5$

Geometry

G2

Identify and classify closed curves in a plane such as a square, rectangle, other parallelograms, triangle and circle

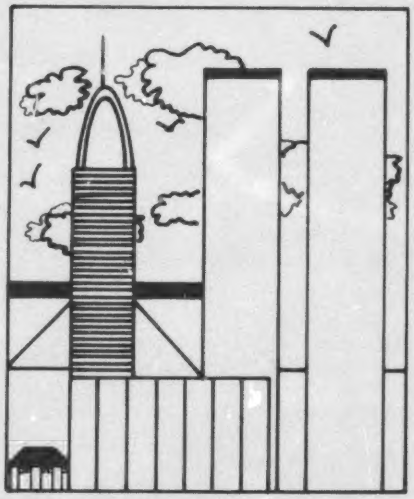
1. Students can be asked to find objects in the school or at their home which have the shape of various closed curves like rectangles, squares, circles, etc. They can be asked to bring pictures (snapshots, magazine pictures, or drawings) for a bulletin board or to share with the class in some other way.
2. Several variations of commercial games can be used to review the identification of closed curves. For example, a rummy game where a card with the drawing ⬠ and a card with the word "pentagon" constitute a pair, could be used. Students can often help prepare these games on index cards that are cut in half. A similar game that could be played with the same set of cards is concentration where the cards are placed face down on a surface and two students try to turn over a matching pair.
3. Classification of quadrilaterals (simple closed curves in the plane which are the union of four line segments) should be introduced by having pupils examine a collection of pictures and observe the following.

   A quadrilateral may have a pair of parallel sides.
   (These are called trapezoid.)

   A trapezoid may have two pairs of parallel sides.
   (These are called parallelograms.)

   A parallelogram may have a pair of perpendicular sides.
   (These are called rectangles.)

   A rectangle may have a pair of adjacent sides which are congruent.
   (These are squares.)



Which shapes can you identify in the skyline?

Some questions which can be raised in a natural way are as follows.

Are all parallelograms similar?

Are all rectangles similar?

Are all squares similar?

After a class has studied classification of triangles, the following questions can be raised.

Are all isosceles triangles similar?
Are all equilateral triangles similar?
Can a right triangle be equilateral?
Can a right triangle be isosceles?

4. Students may enjoy making a mobile or a poster to illustrate some of these relationships:
5. Paper folding activities are excellent to use with this objective. If students are given a square, they can be asked to fold it so that it becomes a triangle, etc.
6. Students with an interest in art will love to analyze paintings in terms of the basic shapes used in the drawing. They may then want to do some drawings to illustrate these ideas.
7. Have a combination of shapes available for students to use creatively to construct pictures. These are particularly nice for holiday pictures such as Christmas trees, Santa Claus, pilgrims, kites, black cats, etc.

Other students could then identify the shapes by name.

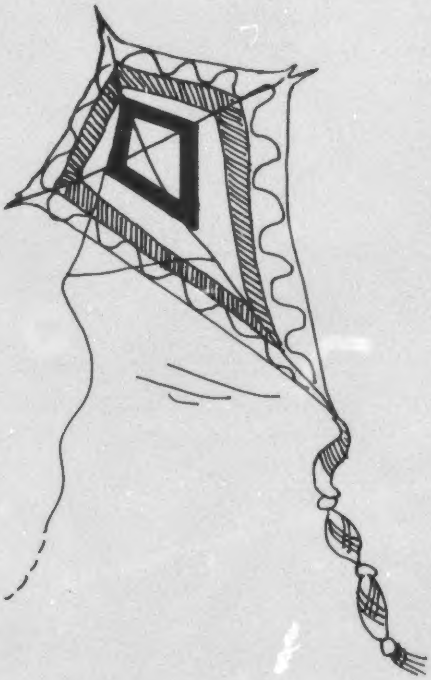
What shapes can you identify in the kite?

Right Triangle

Isosceles

Equilateral

Quadrilateral

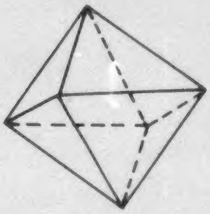


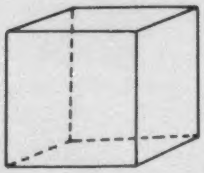
G3
Identify and classify three-dimensional objects such as prism, pyramid, cone and cylinder

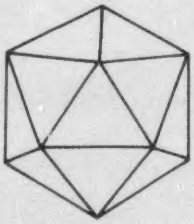
1. The activities suggested for objective 2 may be modified to use with this objective.
2. Students may want to make a booklet containing examples of space figures that they have cut out of magazines, papers, etc.

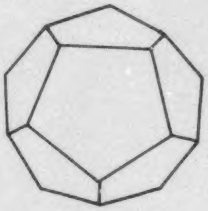
3. The five regular solids are interesting for students to study as an extra project.

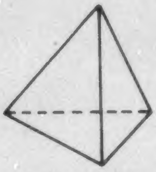
   The five regular solids are as illustrated.


Octahedron


Cube


Icosahedron


Dodecahedron


Tetrahedron

4. An examination of the number of vertices, number of edges, and faces of space figures is an activity that all students may participate in. Euler's formula (F + V = E + 2) where F is the number of faces, V is the number of vertices, and E is the number of edges is an easy one for students to understand after making a chart.

G5
Identify shapes that are alike under rotations (turns), reflections (flips) or translations (slides)

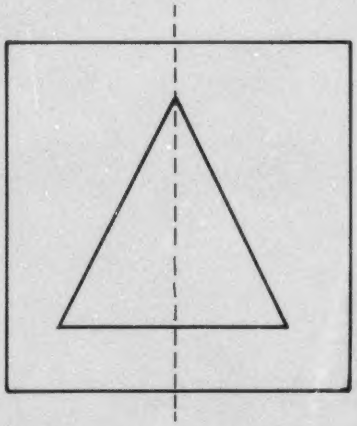
1. Draw a figure on a piece of rubber sheet such as a broken balloon and practice pulling the rubber sheet different ways to see how the appearance of the figure can be changed. Have the pupils sketch several different configurations they can produce from a single drawing as ∞ and ∞ from ∞. Some pupils might like to challenge one another by proposing tricky variations. After some individual experimentations, pupils should be asked to sketch some forms into which the original could not be deformed by pulling. Ultimately the pupils should learn that, for example, ○ can be deformed to △ and □ but not to ■ or to ●, that is, ○, △ and □ are all equivalent. The geoboard is useful to show deformations of simple closed curves. Discussion should then bring out that simple closed surfaces such as cylinder, sphere, cube and the like are all equivalent under deformations.
2. To show the (topological) equivalence of simple paths, direct the attention of the pupils to the way that roads are built from one side of a mountain to the other. Even though the roads may wind around the mountain in different ways, each road goes from one side to the other and does not cross itself. All such roads or paths are topologically equivalent. Have the pupils use a cone and piece of yarn or elastic thread and experiment with some of the different paths they can make.
3. Pupils seem to gain an understanding of similarity from everyday experiences. For example, model cars are similar to actual size ones, and the teacher's writing on the board to illustrate the formation of letters is reproduced similarly by pupils on their papers. Encourage the pupils to supply other examples of equivalent, enlarged or reduced copies.
4. Potato prints can be used effectively to show translation and/or rotation. To illustrate translation, have the pupils make a potato print and then repeat the print. To illustrate rotation, use the same print (or make another) and either turn the potato or the paper each time the print is made.

5. Mirror cards may be used to review or strengthen the understanding of reflection (see reference list of instructional aids). As an additional activity, the pupils may make ink blots and then use either crayons or tempera paint to add to or fill in space on each side of the blot so that one side is a reflection of the other. For further consideration, suggest to the pupils that they see if they can use a potato print to show reflection. They may be surprised to discover that they cannot. Let them use a mirror to check to see if their examples did show reflection. Additional activities may be found in books listed in the annotated references in the media section.

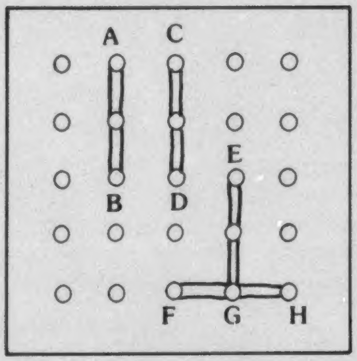
G6

Determine relations between point sets or between geometric figures such as inside, outside, parallel, perpendicular, similar, congruent

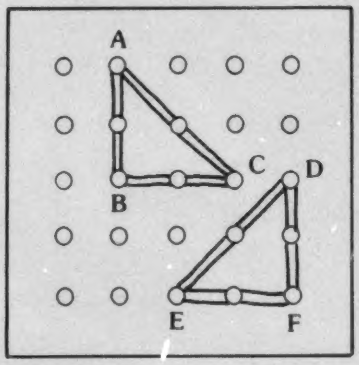
1. Paper folding activities can help students understand the concept of congruent figures. For example, have the students fold an isosceles triangle (a triangle with two congruent sides) so that it forms two congruent triangles.



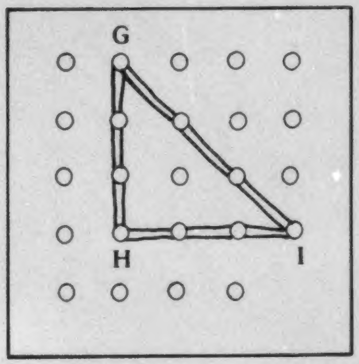
2. Geoboards are excellent for students to be able to illustrate the concepts of parallel, perpendicular, similarity and congruence.



$\overline{AB} \parallel \overline{CD}$
$\overline{EG} \perp \overline{FH}$



triangle ABC is congruent
to triangle DEF



triangle ABC is similar
to triangle GHI

The geoboards are made with nails evenly spaced on a board. Rubberbands are used on the geoboard. A clear one may be purchased for the overhead projector. If geoboards are not available, dot paper may be used.

3. Students can then be given drawings and the teacher can lead discussions concerning the drawings.

   Which pairs of segments in the following figures appear to be congruent?

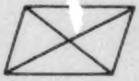


Figure 1

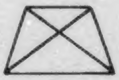


Figure 2

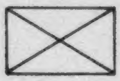


Figure 3

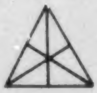


Figure 4

Note: There are 4 pairs of congruent segments in figure 1
4 pairs in figure 2
9 pairs in figure 3
12 pairs in figure 4

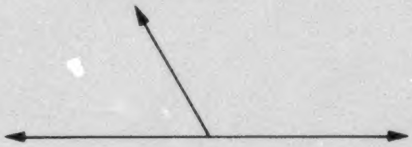
Which pairs of triangles in the figures appear to be congruent?

More mature students should be able to make some generalizations after answering the above questions about figures 1 through 4.

G7

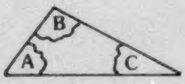
Use relations between and among point sets or geometric figures to deduce other relations, e.g., congruence of alternate interior angles to prove that two lines are parallel

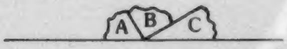
1. After both angle measure and ruler-and-compass construction have been studied some new relations can be identified and summarized.



   a. The measures of the angles in a linear pair sum to 180.

   b. The sum of the measures of the angles of a triangle is 180. This sum can be illustrated by tearing up a triangular disc cut from paper.



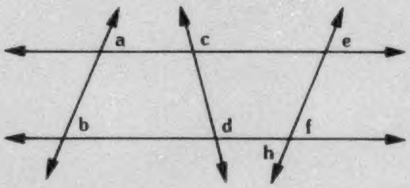


   c. If a triangle has two congruent sides, the angles opposite these sides are congruent. Familiarity with isosceles triangles makes this an obvious observation.

   d. If a triangle has two congruent angles, the sides opposite these angles are congruent. After discussion of c, a question posed to the class should bring out this conclusion.

   e. Two lines in the same plane will be parallel provided that, whenever they are intersected by any third line, a pair of corresponding angles are congruent. The illustration below shows three lines with some corresponding angles identified.

   a and b are a pair of corresponding angles
   c and d are a pair of corresponding angles
   e and f are a pair of corresponding angles
   g and h are also called corresponding angles



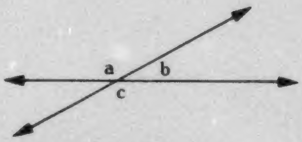
From these relationships, simple arguments could be given for observations already made in exploratory exercises.

Example

f. When two lines intersect, either pair of opposite angles is congruent.

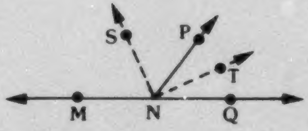
The argument is as follows.

Whatever size ∠a is, the size of ∠b must be such that the numbers sum to 180. (see a. above) Whatever size ∠c is, the size of ∠b must be such that the numbers sum to 180. (again by a.) Thus ∠a and ∠c must be the same size.



g. If both angles in a linear pair are bisected, the bisectors form a right angle.

The measures of ∠MNP and ∠PNQ must sum to 180 (see a. above).
The measure of ∠SNP is half that of ∠MNP.
The measure of ∠PNT is half that of ∠PNQ.
Thus the measures of ∠SNP and ∠PNT must sum to half of 180 or 90.

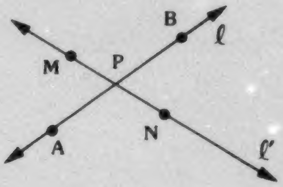


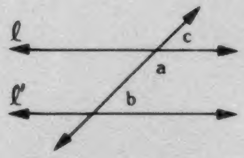
h. Have the pupils bisect a pair of corresponding angles for a pair of parallel lines and note the relation between the bisectors. Then have them give an argument similar to the ones above to show that the bisectors are parallel.

For the pupil capable of moving ahead, some suggested problems are as follows.

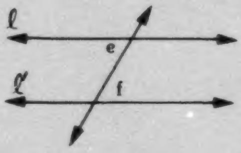
i. Let ℓ and ℓ' be any two lines intersecting at point P. On ℓ, lay off PA = PB; on ℓ' lay off PM = PN. Draw $\overline{AN}$, $\overline{NB}$, $\overline{BM}$, $\overline{MA}$. What figure results? Must this always happen? Try to give an argument to justify your answer.

j. What relationship appears to be true about ∠a and ∠b if ℓ and ℓ' are parallel lines?





k. Give an argument to justify your answer. What relationship appears to be true about ∠e and ∠f if ℓ and ℓ' are parallel lines? Give an argument to justify your answer.



2. Give students sets of regular shaped discs, e.g., a set of triangles that are the same size and shape and have all sides the same length, a set of squares, a set of pentagons with all the sides the same length, etc. Also give the students a set or several sets of irregular polygons (each set contains shapes that are the same size and shape but the lengths of the sides of each disc differ). Ask students to hypothesize whether certain shapes tile or not. Shapes to use might be

   triangles (equilateral, isosceles and scalene)
   quadrilaterals (squares, rectangles, other parallelograms, trapezoids and irregular four sided figures)
   pentagons
   hexagons
   septagons
   octagons

   After students have tried using the different shapes for tiling, ask them *why* and *how* they might predict if a disc will or will not tile. A chart might be suggested if they suggest that the shape will tile if at each intersection the sum of the angles is 360°.

Name of Figure	**Number of Angles**	**Number of Degrees in the Sum of of the Angles**	**Number of Degrees in Each Angle**	**Ability to Tile**
triangle	3	180	60	Yes
quadrilateral	4	360	90	Yes
pentagon	5	540	108	No
hexagon	6	720	120	Yes
septagon	7	900	128 4/7	No
octagon	8	1080	135	No

The discussion of irregular polygons could follow. Some students may need to tile with irregular discs to be convinced. The discussion could lead into one on aesthetics in tiling with regular shapes and lead into combinations of shapes for tiling.

3. The students could be asked to prove that the opposite angles of a parallelogram are congruent by using what they know about angles that are formed when two parallel lines are cut by a transversal.

   Some statements they might use to prove $m\angle 1 = m\angle 4$ are:

$$m\angle 1 = m\angle 8$$
$$m\angle 8 + m\angle 3 = 180^\circ$$
$$\therefore m\angle 1 + m\angle 3 = 180^\circ$$

$$m\angle 3 = m\angle 6$$
$$m\angle 4 + m\angle 6 = 180^\circ$$
$$\therefore m\angle 4 + m\angle 3 = 180^\circ$$

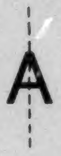
$$m\angle 1 + m\angle 3 = 180^\circ$$
$$m\angle 4 + m\angle 3 = 180^\circ$$
$$\therefore m\angle 1 = m\angle 4$$

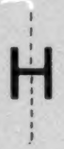
   There are other statements that might be used in a proof of this nature. Students will enjoy exploring with proofs in an informal way.

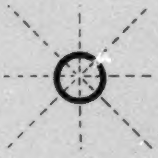
4. The sum of the measures of the angles of a polygon is also an activity that could be used for this objective. This activity can be done by partitioning the polygon into triangles that are drawn by connecting all the diagonals from a given vertex. Once the number of triangles is determined, the students will see that the number of degrees is that number times 180.

G8
Determine lines of symmetry

1. Ink blot pictures are fun for students to use to make their own symmetrical pictures. These pictures can be made by placing a small amount of ink on a piece of white paper and folding the paper. The crease will become the line of symmetry.
2. A collage made of examples of symmetry in nature is a project that most students will enjoy.
3. Various students in the class could be asked to make a study of the alphabet to determine which letters have one axis of symmetry (letter A), two axes of symmetry (letter H), and infinite number (letter O), etc.



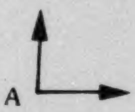


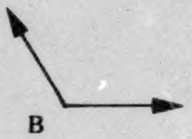


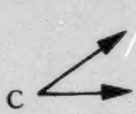
4. Symmetry in sports is another topic of appeal to some students. Lines of symmetry may be drawn on baseball fields, footballs, jerseys, etc.

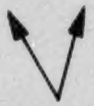
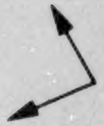
G9
Measure angles and classify as acute, right or obtuse

1. It will be easier for students to measure angles drawn separately prior to measuring angles in drawings. These can be drawn on a ditto and should include angles with various orientations that are different from ∠ A, ∠ B, and ∠ C which have a side parallel to the bottom of the page and the vertex on the left of the drawing.

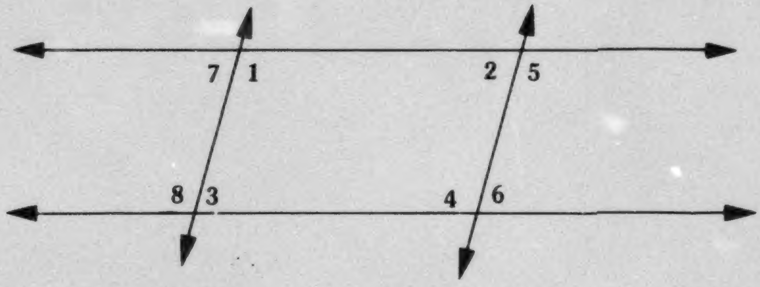






2. Once the students have had experiences with these angles, they can measure angles within drawings. This will enable them to make observations about special drawings, e.g.,
   a) What is the total number of degrees in the three angles of a triangle?
   b) What is true of two of the angles of an isosceles triangle?
   c) How many total degrees are there in the angles of a rectangle? Is this true for all parallelograms?



Instead of using a protractor which has been purchased, pupils can make their own in order to learn how one operates. To make a protractor, have each pupil make several circles on a sheet of paper all of which have the same point as center as shown in figure 1 below.

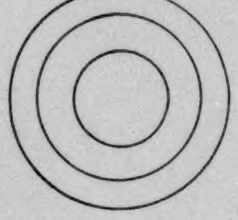


Figure 1

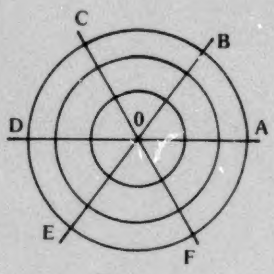


Figure 2

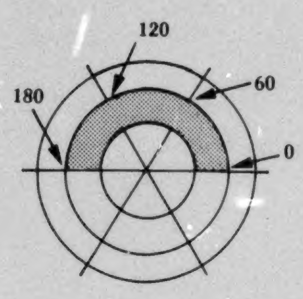


Figure 3

Next, using compasses, partition one of the circles into six congruent parts (figure 2), and draw a line joining each part of subdivision to the center. From previous activities, it should be apparent that, for example, ∠ COB is ∠ DOC rotated, etc., so that six congruent angles are shown. Since these are congruent, numbers should be assigned so that each of these angles can have the same number as its measure. By cutting out the interior of one of the circles, there will be a rim left (shaded in figure 3) on which the numbers can be written. Write O where the ray OA intersects the rim, and 180 where the ray OD intersects the rim. Why 180? Because in our culture that particular assignment has been made: it is arbitrary, as are all of our units, but it is important that the pupils learn the conventional ones. With these numbers written on the protractor the point of rim corresponding to OB should be 60, the one for OC assigned 120. Notice that this will make the measures of each of the congruent angles, ∠ AOB, ∠ BOC and ∠ COD the same; that is 60.

The numbering could, of course, have been started on the left instead of on the right.

By bisecting ∠ AOB, ∠ BOC and ∠ COD points could be located on the rim to be labelled 30, 90 and 150. By cutting out part of the diagram the pupil has his or her own home-made measuring instrument for angles.

G14

Solve simple geometric problems by using properties of similar figures, e.g., indirect measurement

1. Students will need experiences working with simple figures that are similar to learn to use proportions to solve for the missing parts, e.g., in the following figure $\frac{3}{6} = \frac{5}{n}$ or $\frac{3}{5} = \frac{6}{n}$.

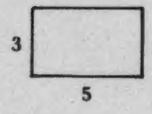


Figure 1

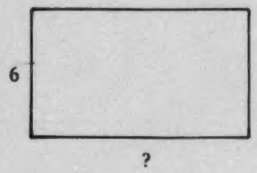


Figure 2

a) The rectangle in figure 1 is 3 units wide and 5 units long. If the rectangle in figure 2 is similar to that in figure 1 and in 6 units wide, how long is it?

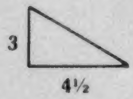


Figure 3

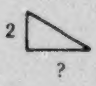
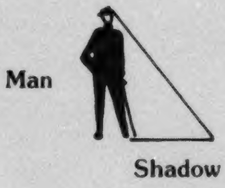


Figure 4

b) The triangle in figure 3 is 3 units tall and 4 1/2 units long. If the triangle in figure 4 is similar to the triangle in figure 3 and is 2 units tall, how many units long is it?

2. Next the pupils might consider people and their shadows.

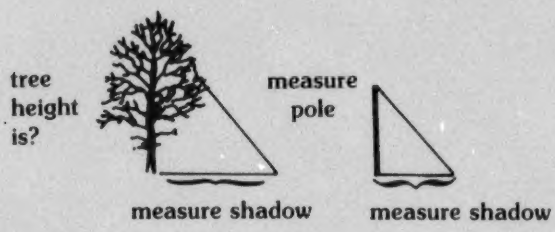


At any given time of the day, shorter people will have shorter shadows. Notice that the shadows are formed by (parallel) rays from the sun, and similar triangles ensue. Using knowledge of the ratios of similar figures pupils can solve the problem. The man in the figure is 6 ft. high and casts a shadow 4 ft. long; if the boy's shadow is 3 ft. long, how tall is the boy?

3. The pupils are probably then ready for the following activity.

Calculate the height of a tree or a building by measuring the shadow.

The children will need metersticks. They are to measure the shadow of the tree or building and also the height and length of the shadow of some accessible object.



From the above measurements, calculate, using appropriate ratios, the height of the inaccessible tree.

The members of the class should work in groups with each group using a different accessible object.

Have them complete this chart.

Class Groups	Object Used	Length of Shadow	Height	Ratio	Length of Tree Shadow	Height of Tree
I	meter stick					
II	fence post					
III	boy					
IV	girl					

Different groups may be expected to arrive at different answers for the height of the tree. Discussion of the discrepancy provides the opportunity to review the approximate nature of measurement. See the strand, Measurement.

Personal Characteristics

This activity is included as an example of a "unit" of mathematics that includes objectives from several of the strands. This cross strand unit may be used in its entirety or several activities from the unit may be used while teaching another unit, e.g., probability and statistics or graphing. The activity is open-ended and contains suggestions for appropriate uses of the calculator in mathematics.

Purpose

To integrate various statistical techniques in an activity that incorporates students' characteristics of eye color and hair color.

Materials Needed

Masking tape or chalk marker

Rope or string

Chalkboard or overhead projector

Calculator (at least one or more)

Objectives

Sets, Numbers and Numeration; Objectives 1, 8, 9, 11, 12, 13, 14

Probability and Statistics; Objectives 1-8, 12, 13, 15

Operations, Their Properties and Number Theory; Objectives 1, 11

Relations and Functions; Objectives 3, 13

Geometry; Objective 9

Processes

Estimating
Translating
Recording
Organizing
Testing hypotheses
Summarizing
Graphing
Judging
Applying
Formulating
Listing
Counting
Defining Problem
Sampling

Procedures

To introduce this activity, have students write two color lists.

Objectives and Processes

Eyes	**Hair**
Blue	Black
Brown	Brown
Green	Red
	Blonde

Estimating

Then (without counting), have them make guesses as to the number of members of the class who have these eye and hair colors.

If students object to the color lists, for example, "Where are hazel eyes?" you can adapt to suit their decisions. Maybe green will become green and hazel.

Organizing
and
Recording
data

P/S
1, 6

Discuss possible methods for determining actual numbers of students who have the hair and eye colors. Students may suggest having each person answer his or her eye color then hair color with a recorder keeping tally marks on a record sheet (a sample is included). Another suggestion may be for all members of the class with certain characteristics to stand (or raise hands), *e.g.*, all blue eyes students stand to be counted. In addition to each student's handout (where they are recording also), have one member of the class record results on the board or overhead projector.

Characteristic Eye Color	Tally	Frequency Number of Students	Relative Frequency	Decimal Equivalent	Percentage
Blue		8			
Brown		17			
Green		3			
Totals		28			

Organizing
data

S/N/N
8, 9, 11-14

O/P/N
1, 11
R/F
3, 13
Translating
data from one
form to another

The chart can be modified to reflect the amount of time you wish to spend and your desired objectives. For example, if you would like to reinforce fractions, decimals and percentages, the last three columns can be completed for eye color and hair color. Any one or more may be omitted if you wish.

Relative frequency is the ratio of the frequency (number of students) to the total number in the group (or class). Representations of these fractions as decimals and then percentages completes the last two columns. It is recommended that calculators be used to find the decimal approximations, and that this number be rounded off to the nearest hundredth and then expressed as a percent. For instance, in this example, 8/28 = .2857142 (the possible display on a calculator) which is rounded off to .29 and expressed as 29%.

In a table completed for a hypothetical class to illustrate this activity, you will notice that the total of the relative frequencies is 28/28 or 1 and the total of the percentage column is 101%. This dismaying fact is due to round-off error and should be pointed out to students and discussed. (The percentage equivalent of one **should** of course be 100%.) Better estimates could be found by rounding the decimal to the nearest thousandth and the corresponding percent to the nearest tenth. You and your class might want to pursue this activity.

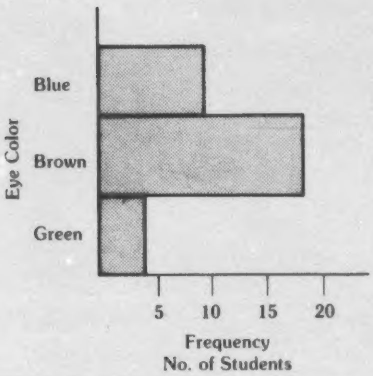
Characteristic Eye Color	Tally	Frequency Number of Students	Relative Frequency	Decimal Equivalent	Percentage
Blue		8	8/28	.29	29%
Brown		17	17/28	.61	61%
Green		3	3/28	.11	11%
Totals		28	28/28	1.01	101%

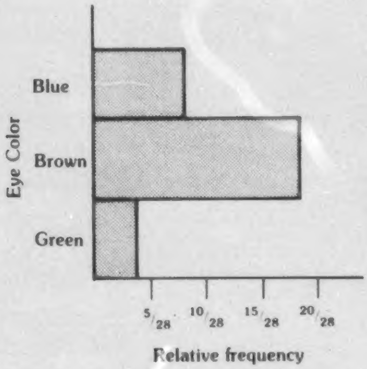
Note: Thus far, the activity could have taken as little as one hour for the higher middle grades or especially bright students or as much as two to three hours for the lower grades or a slower group. Timing really depends on many factors. Do not try to speed up the process. Let the students help make various decisions along the way.

Summarizing

Summarize data from eye color and hair color tables in various formats beginning with a bar graph (histogram). Have students decide how to represent this data. Using the previous example, two of the possibilities for eye color follow.

R/F
13
P/S
6, 7

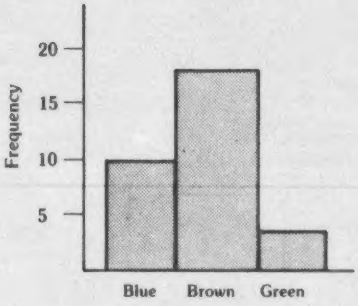


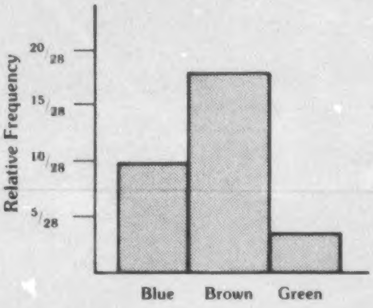


Organizing
Graphing

You might wish to have students consider both to see that the relationships are the same (constant) whether you use frequency or relative frequency.

Another representation of this data follows.





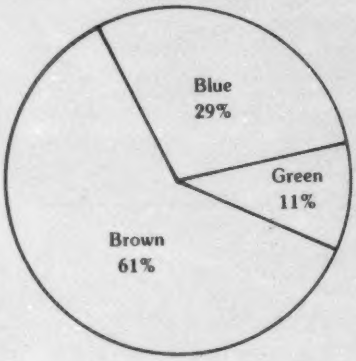
G
9
R/F
13

A circle graph is also a way to summarize the data with corresponding slices of the circle in proportion to the percentages they represent. After determining the appropriate number of degrees, e.g., .29 × 360° = 104°, protractors should be used to draw the graph. A circle graph for the example data follows.

Judging

P/S
6, 7

EYE COLOR



Summarizing

You may want the students to break up into groups and each use one of the graphic methods to summarize the data, and then have the groups present their mode of representation to the whole class.

Analyzing

This portion of the activity can be as brief or as extensive as you choose. The students should at sometime be exposed to various methods of representing such data. If you want to explore this area further, you could follow up with student secured clippings of newspaper articles or magazine articles with graphs of various types. A discussion of the content of the articles could incorporate social studies, science and other subject areas.

S/N/N
1

Formulating
ideas for
collecting
and organizing
data

You may wish to continue this activity with the information already collected. Consider hair color **and** eye color. Can one tell from the tables or graph how many have blue eyes **and** brown hair? Since this information cannot be found in the existing tables, a way of gathering this data needs to be determined. One suggestion is that the students may physically arrange themselves into sets representing various outcomes. The activity can be preplanned by you or evolve with student suggestions. One possible prestrategy would be to have masking tape already on the floor (a large area) in a pattern to represent the four **hair** colors.

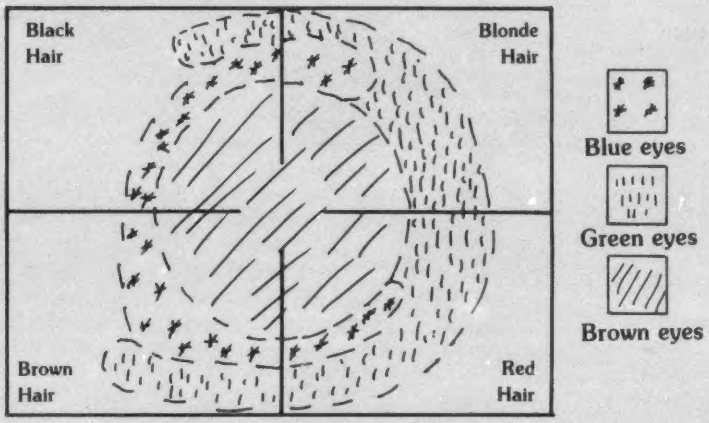
Black	Blonde
Brown	Red

Problem solving

S/N/N
1

Students could be involved in this decision—various patterns are feasible. Ropes, string or chalk could indicate the sets for eye color. Students may experiment but need to eventually determine that possibilities are such that there will be overlap (intersection) for each hair **and** eye color and no overlap **within** hair or eye colors. For example, a person will be in a hair color set **and** an eye color set but will **not** be in two hair color sets.

One possible arrangement might be the following.



P/S
6

Organizing
for data
collection

A jumble of students will probably occur. Students may find that it is too difficult to count themselves using this procedure, you might raise questions such as: How can we describe this data? How many possibilities are there? Have students return to their seats for a discussion of this procedure and possible method of gathering their data.

Note: This activity may become loud and disorganized, but it seems to be justified on the basis of these points.

(a) The students are presented an unclear problem and need to learn to try (and sometimes abandon) various procedures for modeling the problem.

(b) A physical representation of a problem is often a good place to begin.

(c) Students need to use the concept of sets, in this case a Venn diagram, to describe and organize information.

In a discussion of alternative procedures, students can be led to examine the possible pairs of hair and eye color.

Hair Color	**Eye Color**
Brown	Blue
Brown	Brown
Brown	Green
Blonde	Blue
Blonde	Brown
Blonde	Green

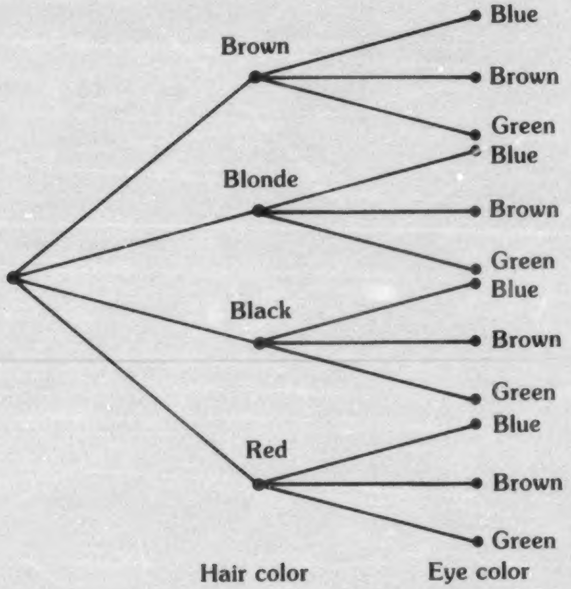
Listing

Black	Blue
Black	Brown
Black	Green
Red	Blue
Red	Brown
Red	Green

P/S
12

To count the pairs, there are **four** hair colors and **three** eye colors or 4 × 3 = 12 pairs. Another possible way to describe this set of outcomes is with a tree diagram.

Counting



Defining
Problem

Judging

P/S
12, 13

At this point, the class will want to know how many of each of the 12 possibilities are represented in their class. Once a list of the outcomes has been determined, i.e., the sample space, the students can determine the "best" way to count themselves. They may decide to arrange themselves again in their jumble or they may decide on a show of hands. Let students record their data for the frequency of each pair of characteristics. You may also want the students to figure the relative frequencies of each pair of characteristics. Decimal equivalents and percentages can also be found as before (with calculators).

Predicting

How can we use this information or data? Can we predict how many people in the school have blue eyes or blue eyes **and** brown hair?

P/S
8
Sampling
P/S
3-5

You can discuss how "representative" the class is as a sample of the school population. If 12% of the students in our class have blue eyes **and** brown hair, can we say that 12% of the school has blue eyes **and** brown hair? Can we somehow **test** our judgment on this?

You can continue further to test the class prediction by taking a poll of the students in the school, either all of them or by taking another sample. These decisions can be made by your class within the constraints of your particular setting. These decisions should be discussed with the class. If for example, there are 300 students in the whole school, you and the class may decide to poll them all. If there are 1,200, you may want to find a sample. How does one choose a sample so that it will be unbiased? How many should be polled? You may wish to use random selection within classes or grade levels, or across the whole school. Let the students make the decisions, they will learn as you point out the possibilities and problems associated with any sampling technique.

A simple poll can be constructed. Let the students decide on the format. It may be a multiple choice poll.

Formulating
Judging

1. Circle your eye color
   a. Blue b. Brown c. Green
2. Circle your hair color.
   a. Brown b. Blonde c. Black d. Red

P/S
1-6

Or they might decide on a fill in the blank format. What possible problems could evolve? Let them find out.

Students must decide how to pass out the polls and how to collect them. They may have to get their material approved by the principal and will need to recruit the cooperation of other teachers.

Students can work in pairs or small groups to record data (probably with tally marks on a sheet similar to the handout). They will be recording hair color and eye color separately and/or the 12 outcomes associated with both characteristics. Some data may be missing due to circumstances such as absent students or classes unavailable due to testing or other reasons. Students must make decisions regarding this problem of missing data encountered in nearly all statistical situations.

S/N/N
12, 13

When all the data is in and recorded, groups of students can calculate relative frequencies and the associated percentages. Once again, encourage your students to use calculators for this task—don't let them get bogged down in arithmetic at this point.

Testing
Hypothesizing
Analyzing
Predicting

Now, they are ready to check the predictive power of their own class as a sample. How close is 12% and 15%? Why is the class a good sample or a poor one? There are many topics for extending the discussion on the predictive power of sampling.

P/S
15

Continuing Evaluation

As such an extended activity progresses, you will have a pretty good idea of who understands various concepts by the level of participation. Questions will be asked and other students can often provide answers or suggestions. You can expect different levels of understanding to have occurred. You might expect that the very bright students will be leading decision makers, e.g., how to record or summarize the data, and what predictions are justifiable. In many situations, this will **not** be the case. Often it is heartwarming to find that the students who seldom otherwise "shine", will be good decision makers and the leaders in initial problem solving settings. All students can participate in the actual data summarization, tallying and so on. All will be able to recognize the various modes of presentation such as the bar graphs and circle graphs. Some students, however, will not be able to make a circle graph with a protractor until they have further experiences. The slower students can be encouraged to participate at their level of achievement. They may be good at verbalizing, but not actually able to construct a graph alone. They may be the ones to explain or tell about it. They may be whizzes with the calculator in processing the data.

Some quiet urging should occur throughout the time span of the activity. Some students will remain uninvolved unless the teacher intervenes to find out what he or she would like to do. This is a group effort and the group dynamics might have to be encouraged if your students are not accustomed to this manner of learning.

At the conclusion of the activity, a summary discussion should be held. Ask questions such as: "What one thing did you like best?" "What did you like least?" "What's one idea you learned?" "How would you change various parts of the activity to make it better?" Let everyone have an opportunity to answer, and take notes so that the students realize you are learning from them. You are also learning how much they absorbed via this activity. By such formative and summative evaluation, you should have a good idea of your students' strengths and weaknesses and what areas to pursue in later activities. Don't expect any of your students to have mastered it all. Expect them to have had different levels of participation and understanding and to have been exposed to mathematics at work in their world.

Further Extensions

This activity could be extended to include the neighborhood or larger community. Sampling techniques and collection of data can become a bit more complicated at this point. The students' enthusiasm (or lack of it) should direct you in this decision whether to proceed or to stop.

If the students have participated in the decision-making, they will have experienced some mistakes and have ideas on how they would change things next time. Another statistical adventure could be planned. Let student interest lead. They may want to poll the class (or school) on their opinion of the lunch room, or how students feel about the upcoming local (or national) election, or how many students own bicycles. If you desire to teach mean, mode, median and range. (Probability and Statistics; Objectives 8, 9, 10) with interval data, you could use age (in months), shoe size, number of siblings in families, or other data that gives you such numbers. The list is endless and so are the challenges.

Task Card Activities

The activities that follow are written in a task card format. These sets of activities are designed in clusters to provide teachers with ideas and student-ready materials. Each cluster is based on a theme which cuts across a wide variety of learning experiences. The three clusters are *Chances Are* — which focuses on objectives for the Probability and Statistics Strand; *Learning Everyday Measurement* — with objectives primarily from the Measurement Strand; and *Ratio, Proportion and the Great Outdoors* — which draws heavily from objectives in the Relations and Functions Strand. The clusters of activities focus on objectives from one strand, but cross over to other objectives when appropriate. This is the nature of the organization of objectives into strands. See the section of this guide "Setting Goals and Objectives — Strands and Objectives."

The activity sets are not to be considered inclusive either for any one objective or for any one strand. Rather, the materials are intended as models or samples for direct use with students and for further development of materials. Teachers may decide to use the task cards as they are, modify them for particular groups or individuals including deletions, adaptations or extensions. Teachers or systems may write additional activities in a similar format.

The format of the three clusters of activities included in this portion of the guide are of Student Cards, Teacher Cards and Parent Cards. Solid lines denote Student Cards, dashed lines denote Teacher Cards and dots indicate Parent Cards. The cards are each 5" by 7" to facilitate their use. They are designed to be copied, cut and possibly attached to 5" by 7" index cards and even laminated if desired. Also, they will fit in a shoe box.

The Student Cards are written directly to the students and often there is space on the cards for their responses. The activities are primarily exploratory and often students will be experimenting as they proceed through the activities.

The Teacher Cards include the objectives for the particular activity, the materials needed, processes that students may use in carrying out the activity, possible modifications for special students, notes to the teacher and sometimes ideas for extensions of the activity. Processes and processing are important aspects of the mathematics curriculum. Wherever possible, teachers should point out to their students the processes involved in their learning. See the section of this guide "Setting Goals and Objectives — Processes." The nature of the middle grade learner and specific needs of individual students are also important considerations for planning activities. The Modifications for Special Students notes on the Teacher Cards are to help teachers plan for students with special needs. For further discussion of these considerations, see the section of this guide "Planning for Instruction." The Notes portion of the Teacher Cards are general comments about content, possible management strategies, questioning techniques, and various other remarks. The teachers, as they consider their class and as they teach the particular lessons, will want to make notes of their own (often on the backs of the Teacher Cards) to facilitate the present lesson and future planning.

If the task cards are used in a school building or system, the teachers involved will want to work together to decide which activities or what variations of the activities will be used for particular groups or levels of students. They should plan so that students will not be asked to work through identical task cards year after year.

Sample Parent Cards are included to help involve the parents in their children's learning. Parents are interested in their young people, but often don't know how to help them. Development of similar Parent Cards is encourgaged. Parents and others can assist in the learning process, help their sons or daughters develop positive attitudes and can also be asked to make materials, etc. See the section of this guide "Planning Instruction — Support Systems."

The activities that follow can assist the teacher in the complex process called planning for instruction. The previous sections of this guide help the teacher understand the students' needs, the societal influences on the schools, the nature of the mathematics being taught and other factors that directly affect the teacher in the classroom. The following clusters of activities are designed for the teacher to help him or her implement instruction and create an atmosphere conducive to student learning.

Chances Are — Probability and Statistics

This unit introduces probability, often called the "mathematics of chance." The students will participate in exploration and experimentation activities based on the following objectives.

Objective		**Remarks**
7	Construct and interpret graphs	Student Card 2
12	Identify the sample space for a simple experiment by (a) describing or tabulating outcomes (b) tree diagrams (c) counting principles	Student Cards 1, 2, 3 and 4
13	Find the probabilities of outcomes including those that are (a) certain to occur (b) certain not to occur (c) equally likely (d) not equally likely	Teacher Card 0 Student Cards 1, 2 and 4
14	Use probability to decide whether or not two outcomes are equally likely	Student Cards 1 and 4
15	Use probabilities to make predictions	Student Cards 2 and 4
16	Find the relative frequency of an outcome in an experiment	Student Card 3
17	Compare predictions based on probability with actual results of an experiment	Student Cards 2 and 4

The following processes are among those used in carrying out the activities.

Generalizing
Applying
Predicting
Experimenting
Verifying

Graphing
Interpreting
Tabulating
Identifying
Comparing

The teacher should make an effort to help the students become aware of the processes they are using. Detailed discussions of processes and processing are found in the "Setting Goals and Objectives" section of this guide.

Probability and Statistics **Teacher Card 0a**

Objective Find the probabilities of outcomes including those that are
(a) certain to occur
(b) certain not to occur
(c) equally likely
(d) not equally likely

Materials A coin

Teacher-led Activity

1. Walk into a class tossing a coin and announce that you have just tossed the coin 19 times in a row and obtained 19 heads. You are about to toss the coin for the twentieth time and you ask the class to predict the outcome. Some students will argue for tails on the basis that it's time for tails to turn up. Others will say that it makes no difference since the probability of heads for any given toss is 1/2, and the probability for tails is also 1/2. After the initial discussion, it may be wise to reinforce the idea that for any toss of a coin (even the twentieth toss), the probability of getting heads, P(H), equals 1/2, and the probability of getting tails, P(T), equals 1/2.
2. Following the discussion, have the students give examples of events that they are certain will happen.

   Example: The sun will set in the west.

Probability and Statistics **Teacher Card 0b**

3. Have students give examples of events that are impossible.

   Example: A dinosaur will walk into the classroom.
4. Have them give examples of events about which there is uncertainity of occurrence.

   Example: It will rain for two weeks from today.
5. Have them give examples of events that are equally likely to occur.

   Example: Getting a 3 or getting a 5 in the roll of a die.
6. Encourage students to discuss the likelihood of two different events for which the chances of occurence are not equally likely.

   Example: It is more likely that an adult will eat bacon rather than peanuts for breakfast.

Notes At this point, you may want to introduce the probabilities of some of the above events or outcomes. The probability of a **certain** event is one and the probability of an **impossible** event is zero. The probability of any other event or outcome is a number between zero and one.

Probability and Statistics **Student Card 1a**

Objectives Identify the sample space for a simple experiment

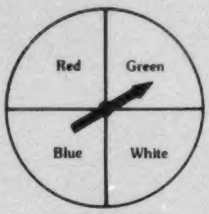
Find the probabilities of outcomes

Use probability to decide whether or not two outcomes are equally likely

Materials None

Activity

1. For spinner I at the right and the **Experiment:** Spin the pointer; identify the **Sample Space** by listing the possible outcomes.

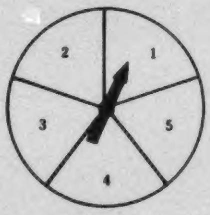


Spinner I

2. How many outcomes are there for this experiment? ___________
3. On a spin of the pointer, the chances you could get Red are 1 out of 4. So the probability of getting Red, P(Red), is 1/4. Find the probability of getting Blue, P(Blue) = ___________.
4. The outcomes (a) getting Red on the spinner and (b) getting Blue on the spinner are **Equally Likely.** Are the outcomes (c) getting White on the spinner and (d) getting Green on the spinner also equally likely? ___________

Probability and Statistics **Student Card 1b**

5. For Spinner II at the right and the Experiment: Spin the pointer; the probability of getting an even number, P(Even), is 2/5 since there are 2 even numbers (2 and 4) and 5 possibilities. Find the probability of getting an odd number, P(Odd) = ___________.



Spinner II

6. Are the events (a) getting an even number and (b) getting an odd number equally likely? ___________

Probability and Statistics **Teacher Card 1a**

Objectives Identify the sample space for a simple experiment

Find the probabilities of outcomes

Use probability to decide whether or not two outcomes are equally likely

Materials None

Processes Generalizing, applying

Modifications for Special Students

Visually impaired students may need verbal instructions and/or descriptions of the spinners if they have problems reading the task cards.

Notes This activity introduces vocabulary and simple concepts like experiment, sample space, outcome, event, probability and equally likely. Definitions of the terms are not given formally, but rather the student is led to understand the vocabulary by example. The simple examples are given and the student will need to apply the language and concepts to answer the questions posed. If this is the first introduction to this vocabulary, you may need to go through more examples with your students.

Probability and Statistics **Teacher Card 1b**

The age and previous experience of your students will help you decide how formal you wish to be regarding the vocabulary. The main goal is to have students understand the terms and to use them appropriately for this and later activities.

Send home Parent Card either before or after this activity.

Probability and Statistics **Student Card 2a**

Objectives Identify the sample space for a simple experiment

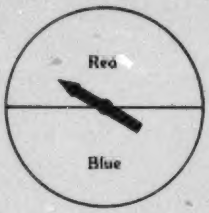
Find the probabilities of outcomes

Use probabilities to make predictions

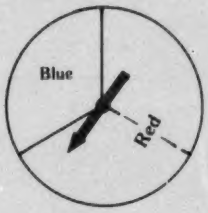
Compare predictions based on probability with actual results of an experiment

Construct and interpret graphs

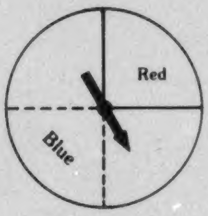
Materials Three spinners



Spinner A



Spinner B



Spinner C

Activity 1. Identify the possible outcomes for each spinner above.

Spinner A ____________________

Spinner B ____________________

Spinner C ____________________

Probability and Statistics **Student Card 2b**

2. Find the probability of getting Red for each spinner. Be careful to consider the **area of the disk** that is Red on each spinner.

   Spinner A: P(Red) = ________ Spinner B: P(Red) = ________

   Spinner C: P(Red) = ________

3. Predict how many times Red and how many times Blue would occur in 12 spins of each spinner and record your predictions (guesses) on Student Card 2c.
4. Spin each spinner 12 times and tally the outcomes on Student Card 2c.
5. Complete the graph on Student Card 2d to show the results of your experiments.
6. Compare your predictions with your actual results for each spinner.

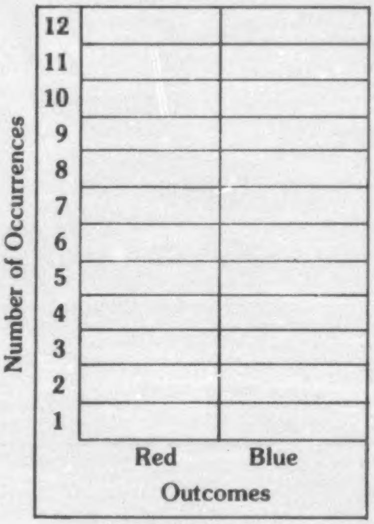
Probability and Statistics **Student Card 2c**

PREDICTIONS AND RESULTS

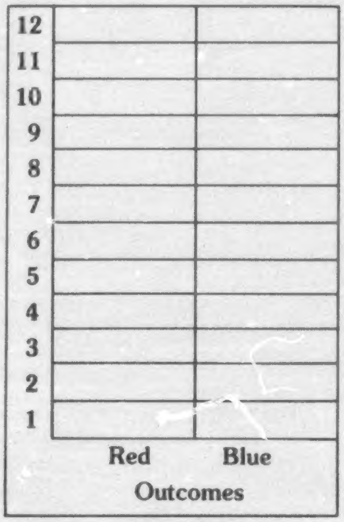
Spinner A		Spinner B		Spinner C	
My Predictions Number Red ________ Number Blue________		My Predictions Number Red ________ Number Blue ________		My Predictions Number Red ________ Number Blue ________	
Actual Results					
Outcomes	Tallies	Outcomes	Tallies	Outcomes	Tallies
Red		Red		Red	
Blue		Blue		Blue	

Probability and Statistics **Student Card 2d**

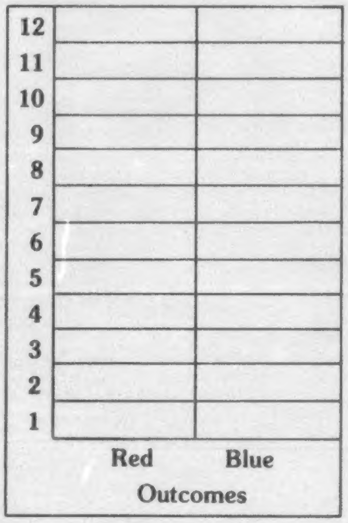
RESULTS OF SPINNER EXPERIMENTS



SPINNER A



SPINNER B



SPINNER C

Probability and Statistics **Teacher Card 2a**

Objectives Identify the sample space for a simple experiment

Find the probabilities of outcomes

Use probabilities to make predictions

Compare predictions based on probability with actual results of an experiment

Construct and interpret graphs

Materials Each student will need three spinners as pictured in Student Card 2a. If you want to make the spinners (or have your students or their parents make them), the following notes may be helpful.

To Make a Spinner

Materials Empty plastic margarine tub or similar container with lid; cardboard or tagboard; marking pens or crayons; glue; brad

How to Trace a copy of a circular disk using the lid of the margarine tub as a template. Mark tagboard as needed for the activity. Glue tagboard to lid of margarine tub and punch a hole in the center of this disk. With brad, attach pointer (made of tagboard) to this disk.

You may wish to place materials for later activities in the tub for storage, e.g., coins, dice and thumbtacks.

Probability and Statistics **Teacher Card 2b**

Processes Predicting, experimenting, verifying, graphing, interpreting, comparing

Modifications for Special Students

- For visually impaired students, modify disks of spinners so that parts of the circular region can be identified by texture rather than color (e.g., velour paper and construction paper).
- For students with motor coordination problems (learning disabled or orthopedically impaired or other) when using disks with spinners, place a rolled piece of masking tape between the underside of the disk and the table or desk top so that the student doesn't have to worry about holding on to the disk. If having the dexterity to spin the pointer is a problem, you might give the student a ruler to activitate the pointer.

Notes Have the students follow the instructions on the Student cards. Require them to predict **before** any actual experimentation. Help create an atmosphere where risk-taking (making their guesses or predictions) is encouraged.

Help students see that the colored areas of the disks for each spinner are different. For example, on Spinner B two-thirds of the area is Red and one-third of the area is Blue. Hence P(Red) = 2/3 and P(Blue) = 1/3. For Spinner C, P(Red) = 1/4 and P(Blue) = 3/4.

Probability and Statistics **Teacher Card 2c**

For 12 trials of the experiment spin the pointer, the students' predictions should be **near** the expected value, e.g., for Spinner B, Red should occur about of 12 or 8 times. Depending on the age and level of your students, you may want to formally discuss the concept of expectation.

Help your students "look back" and compare their predictions with the actual results. It may be that they need to become "better" guessers or it may be that for only 12 trials, their prediction is closer to the expected values than their results seem to indicate. For example, for Spinner A, a student's prediction was 6 Red and 6 Blue and the results were that Red came up 8 out of 12 times. In this instance, the student's guess was better than the results indicate. Explain that probability and predictions based on probability work well "in the long run," i.e., over many many trials.

Encourage students to discuss their results with each other and to compare their findings. You may also want them to pool their data.

Probability and Statistics **Teacher Card 3a**

Objectives Identify the sample space for a simple experiment

Find the relative frequency of an outcome in an experiment

Materials Pairs of dice, paper cups, thumbtacks

Processes Identifying, experimenting, tabulating, interpreting

Modifications for Special Students

- For visually impaired students, the dice may need to be modified so that the student can "feel" the sums. Larger dice with felt dots glued on may be used.
- To help students with motor coordination problems, provide a jar so that the students can use it to "shake out" the dice rather than rolling them.
- Learning disabled students may not have the attention span necessary to tolerate the number of repetitions required for these experiments. You might allow the student to work with one or more other students so that the performance of the task is shared.
- Gifted students may be encouraged to pursue this topic further. Extensions might include making predictions for 100 rolls of the dice and/or 100 tosses of the cup or thumbtack. Pooling data with four other students makes an easy "check" of such predictions. Initiate a discussion of differences between predictions and actual results.

Probability and Statistics **Student Card 3a**

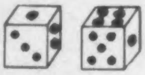
Objectives Identify the sample space for a simple experiment

Find the relative frequency of an outcome in an experiment

Materials Pair of dice, a paper cup and a thumbtack

Activity

1. Roll a pair of dice several times and find the sum of the two numbers on the upper faces each time. Identify all possible outcomes.



2. Roll the pair of dice 36 times. Sum the numbers on the two upper faces each time and tally the results on Student Card 3c.
3. After you've completed the 36 rolls, count up the tally marks and record this number as the frequency for each outcome. Also record the relative frequency, that is, the ratio of the frequency to total rolls.
4. Answer the following questions for the experiment.

Which sum(s) seem to be most likely? Why?

Which sum(s) seem to be least likely? Why?

Probability and Statistics **Student Card 3b**

5. Think about tossing a thumbtack. What are the possible outcomes?
6. Actually toss a thumbtack 20 times and record your results in Chart I on Student Card 3d.

7. Answer the following questions for the experiment.

Do the outcomes seem to be equally likely?

Which outcome seems to be most likely?

Which outcome seems to be least likely?

8. Toss a paper cup 20 times to see if it lands with the open end up, the bottom end up, or on its side. Record your results in Chart II of Student Card 3d.
9. Answer the following questions for the experiment.

Do the outcomes seem to be equally likely?

Which outcome seems to be most likely?

Which outcome seems to be least likely?

Probability and Statistics **Student Card 3c**

36 ROLLS OF TWO DICE

Sums	Tally	Frequency	Relative Frequency
2			/36
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
Totals		36	

Probability and Statistics **Student Card 3d**

Chart I 20 TOSSES OF A THUMBTACK

Outcomes	Tally	Frequency	Relative Frequency
Point straight up			/20
Point straight down			
On its side			
Totals			

Chart II 20 TOSSES OF A PAPER CUP

Outcomes	Tally	Frequency	Relative Frequency
Open end up			/20
On its side			
Bottom end up			
Totals			

Probability and Statistics **Teacher Card 3b**

Notes You may want to consider letting students perform these experiments in pairs. Working together helps alleviate the possible boredom of such repetitions, allows for some conversation and speculation as the experiment progresses and in most cases is more fun for the students.

Discussion of impossible events could arise if "thumbtacks: point straight down" is considered and you are not tossing tacks on carpet.

When the students total their frequencies in their Tables, they should get the total number of trials. Similarly, when they total their relative frequencies, they should get **one.**

Extension Have the students identify the complete sample space for rolling two dice. A chart is a good way to describe the possible ordered pairs of dice.

Considering two dice of different colors will help the students distinguish between the dice, e.g., getting a 4 on the red die and getting a 2 on the green die (4, 2) is different from getting a 2 on the red die and getting a 4 on the green die (2, 4), although both sums are 6.

Have students find probabilities for each sum, e.g., P (a sum of 6) is 5/36 since there are 5 ways to get a sum of 6 and there are 36 possible outcomes. See the Chart on Teacher Card 3c. The diagonal marked is where sums of 6 are found.

Probability and Statistics **Teacher Card 3c**

After students determine the probabilities for various sums, have them make predictions, carry out experiments and then compare their predictions with actual results.

SAMPLE SPACE FOR ROLLING TWO DICE

Green Red	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Probability and Statistics **Student Card 4c**

Chart I 20 TOSSES OF ONE COIN

Outcomes	Tally	Predicted Total	Actual Total
Heads (H)			
Tails (T)			

Chart II 20 TOSSES OF TWO COINS

Outcomes		Tally	Predicted Total	Actual Total
Penny	Dime			
H	H			
H	T			
T	H			
T	T			

Probability and Statistics **Parent Card**

Objectives Become aware of the part probability and statistics play in our daily lives.

Activity

1. Discuss how probability and statistics are a part of our daily lives: "I will probably be home by 3:00." "It probably won't snow tomorrow." "The average rainfall for July was three inches.
2. Make students aware of the ideas of probability and statistics in newspapers, magazines, or on television.

   Weather reports such as 30 percent chance of rain
   Polls or questionnaire results.
   Graphs in magazines and newspapers as summaries of data.
   The "statistics" of sports — e.g., football, baseball, etc.
   Advertising claims — e.g., two out three dentists recommend . . . .

Probability and Statistics **Teacher Card 4a**

Objectives Identify the sample space for a simple experiment.

Find the probabilities of outcomes.

Use probability to decide whether or not two outcomes are equally likely.

Compare predictions based on probability with actual results of an experiment.

Materials A penny and a dime for each student.

Processes Predicting, experimenting, interpreting, comparing.

Modifications for Special Students

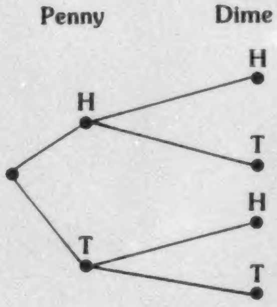
- Motor coordination problems can be alleviated by using a jar so students can "shake out" the coins as they did with dice. (see Teacher Card 3a).
- Short attention spans - also see Teacher Card 3a — you may want students to work in pairs.
- Extensions for gifted students might include predictions for 1000 or more coins. Caution: do not ask student to check predictions by tossing 1000 coins. If a check is desired, further pooling of data is recommended or computer simulation of coin tosses. Other extensions might include similar Student Cards for tossing three or four coins. You might have gifted students design such cards for later presentation to the whole class.

Probability and Statistics **Teacher Card 4b**

Notes See Teacher Card 3b for comments on students working in pairs and later pooling of data. These earlier remarks are also appropriate for this activity.

Encourage students to identify the sample space for the experiments by listing the outcomes and you may also introduce tree diagrams. For example, for the experiment toss two coins, a listing or tabulating of the outcome would be

$S = \{HH, HT, TH, TT\}$



A tree diagram that describes the sample space is shown to the right.

Encourage discussion of differences between students' predicted results and actual results. Point out that probability and expectation work well over many trials, i.e., "in the long run." For example, the probability of getting 2 Heads for the toss of 2 coins is 1/4. For 100 trials, one would **expect** that for about 1/4 of 100 or for 25 trials, both coins would be Heads. However, actual results might be 23 out of 100

Learning Everyday Measurement

This unit is designed to provide concrete measurement of experiences whereby middle grade students can develop an understanding of measurement using nonstandard and standard units. These concrete measurement experiences include the measurement of time, length, perimeter, mass, area, volume and capacity. The following objectives are included in this unit.

Objective		**Remarks**
Measurement		
1	Determine a time interval between two events	Student Card 2
3	Determine the mass (weight) of an object using (a) nonstandard units and (b) standard units	Student Card 4
4	Make a reasonable estimate of mass (weight) of an object or substance and verify the estimate	Student Cards 4, 5
5	Determine capacity or volume by counting (a) nonstandard units and (b) standard units	Student Card 6
7	Make a reasonable estimate of capacity or volume of a rectangular container and other three-dimensional objects and verify the estimate	Student Card 7
11	Apply the formula for the area of a rectangular region to derive formulas for the area of other regions	Student Card 8
13	Determine a length or distance using (a) nonstandard units and (b) standard units	Student Cards 1, 2
14	Make a reasonable estimate of a length or distance and verify the estimate	Student Card 3
15	Derive formulas for finding perimeters of simple closed curves	Student Card 10
16	Select the appropriate type of measurement needed for a given problem situation, e.g., length, area, volume	Student Card 9 Teacher Card 11
17	Select the appropriate formula(s) for finding the measurement for a given situation, e.g., area, perimeter, circumference, volume	Student Card 9 Teacher Card 11
Geometry		
10	Use formulas to solve geometric problems involving perimeter, area and volume	Teacher Card 11
Probability and Statistics		
9	(Partial) Find the mean of a set of numbers	Student Card 10

The following processes are among those used in carrying out the activities.

Estimating
Measuring
Recording
Comparing
Cooperating
Verifying
Looking back
Interpreting
Deriving
Using Resources
Constructing
Problem focusing
Computing with a calculator
Averaging
Generalizing
Modeling
Scaling
Decision making
Approximating
Problem finding

The teacher should make an effort to help the students become aware of the processes they are using.

Detailed discussions of Processes and Processing are found in the "Setting Goals and Objectives" in this section of this guide.

Measurement **Student Card 1a**

Objective Determine length or distance using (a) nonstandard units and (b) standard units.

Materials String, scissors and some paper clips of the same size

Activity

1. Estimate how many hands long the teacher's desk is. Record your estimate in the table on Student Card 1b. Measure the desk with your hands and record the actual measurement on the Student Card.
2. Continue this procedure by measuring other objects such as the chalkboard height, width of the room, window sizes, length of the playground, etc.
   * For each object estimate and record your estimate. Then measure that object and record this actual measurement. Use your experience to help you become better at estimating.
3. Measure ______________________ 's height with your hands.
   (some student in the room)

   Use your hand measurement to cut a length of string that is exactly as tall as the student you measured.
4. Make a paper clip chain in segments of 10. Estimate and then measure the same objects you measured in number 2 using the paper clip chain. See * above.

Measurement **Student Card 1b**

TABLE OF VALUES

Object	Number of Hands		Number of Paper Clips	
	Estimate	Actual	Estimate	Actual
1. Teacher's Desk				
2.				
3.				
4.				
5.				
6.				
7.				
8.				
9.				
10.				

Measurement **Teacher Card 1a**

Objective Determine length or distance using (a) nonstandard units and (b) standard units

Materials Several boxes of the same size paper clips, a ball of string, scissors

Processes Estimating, measuring, recording, comparing

Modifications for Special Students

- Visually impaired students may need the opportunity to **feel** the object before an estimate is made.
- Orthopedically impaired students may need to be paired with an unimpaired student to facilitate manipulation of paper clip chains.
- Health impaired students might need to be the **Official Recorders** of statistics for either a small group or the entire class when students are involved in some of the more vigorous activities such as measuring the length of the playground.
- Gifted students may not need to measure as many objects as other students. As their estimates become fairly close to actual measurements, direct them to other activities. A report or poster on the history of measurement or a particular system of measurement may be appropriate.

Measurement **Teacher Card 1b**

Notes Have students continue measuring objects until their estimates are about the same as their actual hand measurements.

When choosing one student for everyone to measure with their hands, the nature of the middle grades learner will help you make your choice. Probably one near the average height would be best.

After measuring, each student will cut a string based on his or her hand measurement. Hang all of the strings from a standard position, perhaps the top of the chalkboard, and notice that the strings are not the same length. Discuss with the students the need for having something of the same size as a measuring unit.

When measuring with the paper clip chains, students should determine nearly the same measurements for a particular object. A comparison of the measuring devices, hands and paper clips, should lead the students to conclude that paper clips are a much more accurate standard measuring tool than the hands of individual students.

Rather than having students work individually on this task, you might suggest that pairs of students work together.

Measurement **Student Card 2a**

Objective Determine a length or distance using standard units

Determine a time interval between two events

Materials Several yard sticks and meter sticks and a stopwatch

Group Activity

1. You will participate in a softball throwing contest, a standing broad jump contest and a 100-meter dash.
2. Measure the length of your softball throw and the length of your jump in yards and feet. Also measure each length in meters and centimeters. Record this data on Student Card 2b.
3. You will run the 100-meter dash and have someone record how many minutes and seconds it took you to run this distance. Record this data as indicated on Student Card 2b. You will also time someone else.
4. Notice any relationships between the lengths in English units and metric units.

Measurement **Student Card 2b**

MY RECORDS

I threw the softball ______ yds. ______ ft.
______ m ______ cm } length

I can jump ______ yds. ______ ft.
______ m ______ cm } length

I can run 100 meters in ______ min. ______ sec. } time

Measurement **Teacher Card 2a**

Objective Determine a length or distance using standard units

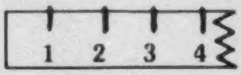
Determine a time interval between two events

Materials Several yard sticks, meter sticks and a stopwatch

Processes Measuring, recording, comparing, cooperating

Modifications for Special Students

- Orthopedically impaired and health impaired students may need to be **Official Recorders** if they cannot participate in the physical activities.
- Hearing impaired students may need a visual signal from the timer when running the 100-meter dash.
- Visually impaired students may need to **feel** the marks for various units. Along each foot mark on a yard stick and for each centimeter on a meter stick, squirt a thin line of glue and sprinkle with sand.



A parent may be called upon to help make these materials.

Notes Conduct your own version of the Olympics by having students participate in a softball throwing contest, a standing broad jump and a 100-meter dash.

Measurement **Teacher Card 2b**

Mark off the starting lines for each contest. Let a couple of students lay out the 100-meter race track.

Have students rotate through the events with approximately one-third of the students at each event at any one time. If you have a large number of students, consider two areas for each event, i.e., six locations.

The students will measure the length of their throws and the lengths of their jumps in yards and feet, e.g., 3 yards 2 feet and in meters and centimeters, e.g., 3 meters 35 centimeters. Do **not** have students struggle with conversions between metric and English units. However, comparisons should be encouraged, e.g., 3 yards is nearly 3 meters.

For the 100-meter race, in order to assure that each student has an opportunity to use the stopwatch, pairs of students might be asked to time each other.

If desired, you can award prizes or ribbons to first, second and third place winners of each event.

Extensions An extension of this activity might include collecting group data and finding the average or mean for the softball throw and/or the broad jump. You might also find the mean or average time for the 100-meter dash.

An activity of constructing graphs is also possible. Be careful, however, not to let students directly compare individuals with the group or other individuals.

Measurement **Student Card 3a**

Objective Make a reasonable estimate of length and verify the estimate

Materials Metric tape measure

Activity You will need to work with a partner for this activity

1. Both you and your partner estimate your own head size in centimeters. Write your estimate on Student Card 3b. Then take turns. Measure your partner's head and have your partner measure your head. Record the actual length to the nearest centimeter on Student Card 3b.
2. For each of the other body parts on Student Card 3b, estimate and then measure. Try to measure carefully and use each measurement to help sharpen your estimation abilities.
3. After you complete Student Card 3b read and answer these questions.

   Often people use the distance around their fist to estimate sock size. Have you ever gone shopping, picked up a sock and wrapped it around your fist to see if it will fit your foot? _________ Do your measurements seem to indicate that this is a good way to estimate your sock size? _________ What other body part seems to be approximately the same size as your foot? _________

Measurement **Student Card 3b**

MY VITAL STATISTICS

Body Part	Estimate	Actual Measure
Head size	_________ cm	_________ cm
Wrist size	_________ cm	_________ cm
Ring finger size	_________ cm	_________ cm
Fore-arm length (from wrist to elbow)	_________ cm	_________ cm
Fist size (distance around fist)	_________ cm	_________ cm
Foot size	_________ cm	_________ cm

Look for hidden relationships among the actual measures. For example

My head size is about _________ times my wrist size.

Measurement **Teacher Card 3a**

Objective Make a reasonable estimate of length and verify the estimate

Materials Metric tape measures

Processes Estimating, measuring, comparing, cooperating with others

Modifications for Special Students

- Visually impaired students may need a modified tape measure similar to the yard stick or meter stick described in Teacher Card 2a.
- Gifted students may be encouraged to measure to the nearest millimeter or tenth of a centimeter. Later they may need to round off to make comparisons.

Notes Students will need to work in pairs for this activity.

Encourage students to measure carefully.

Estimates should be made **before each** measurement. If the students make all of the estimates first, then find all of the measurements, they will not have any feedback to make *better* estimates each time.

Measurements of the students' feet should be made with their shoes off. Later comparisons will be better.

Fist size should be approximately the same as foot size. Also, the length of the forearm is near the foot size.

Measurement **Teacher Card 3b**

Encourage students to look for other relationships among sizes of body parts. For example, head size will be about four times wrist size and three fingers is near one wrist.

Extensions You might want students to find the mean, median, mode and range for some of this data. Wrist size is a good measure to choose since nearly all wrists will be between 13 centimeters and 15 centimeters and comparisons among students should not prove to be too sensitive. Fore-arm length will probably be more variable and is also a measure that will be fine for comparisons. Actual foot size may be a sensitive measure for some students and for this reason, avoid size comparisons of feet.

Measurement **Student Card 4a**

Objective Determine the mass (weight) of an object using (a) nonstandard units and (b) standard units

Make a reasonable estimate of mass (weight) of an object and verify the estimate

Materials A balance scale, metric masses, metal or plastic hardware washers of the same size, and items such as: a stick of gum, a nickel, a baseball, a chalkboard eraser, a pencil, a mathematics book

Activity

1. Determine how many washers are equal in mass to each item by using the balance scale. Record your results on Student Card 4b.
2. Use the standard metric masses and the balance scale to determine each item's metric mass and record these results on Student Card 4b.
3. Consider the data in your chart and make an estimate of the mass of **one** washer.
   I think **one** washer is about _________ grams.
4. Use the metric masses and the balance scale to verify your estimate.
   The actual mass of **one** washer is _________ grams.

Measurement **Student Card 4b**

RECORD OF MASS

Item	Number of Washers	Metric Mass
1.		
2.		
3.		
4.		
5.		
6.		
7.		
8.		
9.		
10.		

Measurement **Teacher Card 4a**

Objectives Determine the mass (weight) of an object using (a) nonstandard units and (b) standard units

Make a reasonable estimate of mass (weight) of an object and verify the estimate

Materials Balance scale, metric masses, metal or plastic hardware washers of the same size, and items such as a stick of gum, a nickel, a baseball, a chalkboard eraser, a pencil, a mathematics book

Processes Measuring, comparing, estimating, verifying

Modifications for Special Students

- Visually impaired students may need verbal instruction with the metric masses so they can **feel** them and know the particular comparisons. They may not be able to read the labels such as 5 grams so may need to hold the 5-gram mass and five unit gram masses and be told that they are equivalent. Special markings on the mass are not recommended since this would change the standard mass itself.

Notes Substitutions can be made for any of the items in the materials list except the balance scale and metric masses. If washers of the same size are hard to find, you might use paper clips or some other nonstandard mass such as thumb tacks.

Measurement **Teacher Card 4b**

Using nonstandard masses helps convey the idea of measuring and helps students see the need for some agreed upon standard mass. Discuss these ideas with the students. These measurements could also be done with ounces and pounds — another standard measuring unit.

Comparisons of the nonstandard mass, the washer and the standard metric mass may simply be made by guessing or estimating. You could later point out how to use ratios and proportions to estimate. For example, if an item is found to be 4 washers and 8 grams, then this is a 1 to 2 ratio, hence the mass of 1 washer would equal 2 grams.

Measurement **Student Card 5a**

Objective Make a reasonable estimate of mass (weight) of an object and verify the estimate.

Materials You will need a balance scale, metric masses and various objects of your choosing.

Activity

1. Pick up a one gram mass in one hand. Then pick up several objects one at a time with the other hand until you find something you think has the mass of one gram. Record the name of this object in Chart I on Student Card 5b.
2. Use a balance scale to measure the actual mass of the object and record its mass in Chart I on Student Card 5b.
3. Find two more objects you estimate have a mass of one gram and record the names of these objects in Chart I. Then find and record their actual masses.
4. Repeat the experiment using a kilogram mass and record this data in Chart II on Student Card 5b.
5. Choose two other objects. This time, estimate their masses and record the objects' names and your estimates of their masses in Chart III on Student Card 5b. Use the balance scale to verify your estimates and record the actual masses in Chart III.

Measurement **Student Card 5b**

ESTIMATING MASS

Chart I

Objects estimated to have mass of one gram	Actual Mass
1.	
2.	
3.	

Chart II

Objects estimated to have mass of one kilogram	Actual Mass
1.	
2.	
3.	

Chart III

Other Objects	Estimated Mass	Actual Mass
1.		
2.		

Measurement **Teacher Card 5a**

Objective Make a reasonable estimate of mass (weight) of an object and verify the estimate.

Materials A balance scale, metric masses, objects of varying masses. (Objects that may be close to a mass of one gram are paper clip, thumb tack, pencil stub, life saver, small piece of chalk, round toothpick, small plastic washer. Objects that may be near one kilogram in mass include man's shoe, large cabbage, one liter of water, large book such as a dictionary. Other objects should also be available and mixed in with those you know may be near one gram or one kilogram.)

Processes Estimating, measuring, verifying, comparing

Modifications for Special Students

- Visually impaired students may need a reminder about the equivalences of various metric masses. See Teacher Card 4a.

Measurement **Teacher Card 5b**

Encourage students to *become* the balance scale pictured on Student Card 5a. They will begin to **feel** the masses — one gram and one kilogram.

At the conclusion of this activity, discuss the relationship between one gram and one kilogram. Encourage student participation. Ask questions such as "How many of these (gram) masses do you think it would take to equal one of these (kilogram) masses?" "How could we find out?"

Since you will probably not have and could not fit 1000 gram masses on the balance scale, have students suggest how they can verify their guesses. Other masses of 5 grams, 10 grams, 100 grams, etc. could be used to balance the one kilogram mass. Using this demonstration will be impressive and help students remember the relationship between gram and kilogram.

Measurement **Student Card 6a**

Objective Determine capacity or volume by counting (a) nonstandard units and (b) standard units.

Materials Small box, marbles, small balls of cotton, small triangular solids, small rectangular solids, sugar cubes and models of cubic centimeters and cubic inches

Activity

1. Guess the number of marbles needed to fill your box and record this estimate on Student Card 6b.
2. Fill the box **completely** full of marbles.
3. Count the number of marbles used and record your result on Student Card 6b.
4. Repeat the procedure using the balls of cotton, the triangular solids, the rectangular solids and finally the sugar cubes.
5. Which materials were best suited to use to find the amount of space or volume of the box? Why?
6. Repeat the procedure using standard units to measure volume — the cubic centimeter and the cubic inch. Be sure to estimate first, and then count.

Measurement **Student Card 6b**

VOLUME OF A SMALL BOX

Measuring Unit	Number needed to fill the box Estimate	Actual
1. Marbles		
2. Cotton Bails		
3. Triangular Solids		
4. Rectangular Solids		
5. Sugar Cubes		
6. Cubic Centimeters		
7. Cubic Inches		

Measurement **Teacher Card 6a**

Objective Determine capacity or volume by counting (a) nonstandard units and (b) standard units

Materials Small boxes, marbles, small balls of cotton, small triangular solids, small rectangular solids, sugar cubes and models of cubic centimeters and cubic inches

Processes Estimating, comparing, looking back, measuring

Modifications for Special Students

- Visually impaired students will need to **feel** the boxes and the measuring units before making estimates.

Notes You might want students to work on this activity in pairs or small groups.

Discuss the advantages and disadvantages of the nonstandard measuring units. For example, marbles leave spaces which are not filled and therefore have a disadvantage as a unit for measuring volume.

An intuitive comparison of cotton balls and marbles or cubic centimeters and inches could be considered. Do not, however, ask students to convert from one unit to another.

Measurement **Teacher Card 6b**

Extension Objective: Use experimentation to derive the formula for volume of a rectangular container.

Lead a discovery lesson for finding volume of a rectangular solid. Have the students use the same box and measure the length, width and height in centimeters. Encourage them to look for a relationship between these three measures and the volume in cubic centimeters — the number of cubic centimeters used to fill their box. Groups of students with boxes of varying sizes should compare their results. Have them generalize their findings and make conjectures (guesses) for volume of other rectangular solids. The formula $V = l \bullet w \bullet h$ will be the general result.

Measurement **Student Card 7a**

Objective Make a reasonable estimate of capacity of a rectangular container and other three-dimensional objects and verify the estimate

Materials Cup, half-pint milk carton, masking tape, a glass jar, pint and quart containers, empty cola or juice can, an ice-cube tray and a cone-shaped paper cup

Activity

1. Guess the capacity to the nearest milliliter of the cup. Record this estimate on Student Card 7b.
2. Take the half-pint milk carton and make a mark two centimeters from the bottom. Pour water up to this mark. This capacity is about 100 millimeters.
3. Pour the 100 millimeters of water into the glass jar. Place a strip of tape on the jar to mark this level. Use the milk carton to help mark these levels: 220 millimeters, 300 millimeters, 400 millimeters and 500 millimeters.
4. Use the jar to find out the approximate number of milliliters in the cup and record this capacity on Student Card 7b.
5. Repeat the procedure of estimating and then measuring for the other containers in the materials list. Be sure to estimate first and record this guess, then measure.

Measurement **Student Card 7b**

ESTIMATING CAPACITY

Container	Estimate	Actual Capacity
1. Cup		
2.		
3.		
4.		
5.		
6.		
7.		
8.		
9.		

Did your estimate get better as your experience increased? __________

(If your answer above is NO, you need some more experience in estimating.)

Measurement **Teacher Card 7a**

Objective Make a reasonable estimate of capacity of a rectangular container and other three-dimensional objects and verify the estimate

Materials Cup, half-pint milk carton, masking tape, a glass jar, pint and quart containers, empty cola or juice can, an ice-cube tray and a cone shaped paper cup (Substitutions can be made for many of these containers)

Processes Estimating, measuring, verifying, looking back

Modifications for Special Students

- Visually impaired students may need the measuring instrument modified so that they can **feel** the 2 centimeter mark. See Teacher Card 2a. They may also need a student partner to help calibrate their glass jar.
- Orthopedically impaired students may need help in calibrating and pouring the water. A funnel may be used when pouring into a small mouth container.
- Gifted students may be encouraged to investigate the relationships between capacity, mass and volume in the metric system. For distilled (pure) water; one milliliter has a mass of one gram and a volume of one cubic centimeter. A liter of pure water would have a mass of one kilogram and would fill a cube that is 10 centimeters on each side.

Measurement **Teacher Card 7b**

Notes Explain that it takes 1000 milliliters to equal one liter. You might like to have a prepared container with one liter of water to show the students before they begin the activity. A liter bottle of soft drink may be a good example.

A common measuring cup contains approximately 250 millimeters of water. Do not expect the students to convert from one measuring system to another, but general comparisons are all right. For example, one liter is a little more than one fluid quart.

If you have access to calibrated cylinders, you might have students use these in lieu of Steps 2 and 3 on Student Card 7a. Their measurements will be more accurate.

Variation After students have estimated and then measured the capacities of several containers, you could continue this activity as a game.

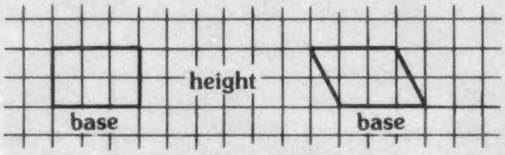
Separate the class into two teams and have each team make an estimate of the capacity of a particular container. Then have a student from each team measure the capacity of the container. The team having the better estimates most often wins the game. You should plan for students to measure an odd number of containers to avoid ties.

Measurement **Student Card 8a**

Objective Apply the formula for the area of a rectangular region to derive formulas for the area of other regions

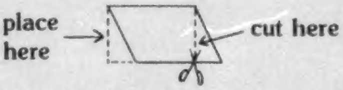
Materials Grid paper and scissors

Activity 1. On a sheet of grid paper draw a rectangle that is 3 squares long (its base) and 2 squares high (its height). Draw a parallelogram that is not a rectangle that also has a base 3 squares long and a height of 2 squares.



2. Find the area of each region using your grid paper to help you count square units.

3. Cut out each region. Then cut the parallelogram region as shown below to make a rectangular region. Fit the pieces of the parallelogram region over the rectangular region.



Measurement **Student Card 8b**

4. Notice any relationships between the area of the parallelogram region and the rectangular region.

5. Repeat this experiment varying the size of the regions. For example, choose a rectangular and parallelogram region both having a base of 5 and a height of 4.

6. Use the results of the experiment to help you write a general formula for the area of a parallelogram region: A = _________.

7. Test your formula on several other parallelogram regions. Are you convinced that it will always work? If not, modify your formula or make more tests.

Measurement **Teacher Card 8a**

Objective Apply the formula for the area of a rectangular region to derive formulas for the area of other regions

Materials Grid paper and scissors

Processes Constructing, interpreting, comparing, deriving

Modifications for Special Students

- Gifted students may be led to derive areas of other regions such as that of a trapezoid or other nonregular quadrilateral.

Notes If students are not familiar with the area of a rectangle, precede this activity with other activities that will enable students to make this discovery. See Measurement Objective 10.

Areas of both the initial rectangular and parallelogram regions will be 6 square units. When the student cuts the parallelogram region, the pieces should fit over the rectangular region and cover it completely. Be sure the students see that the areas are exactly the same.

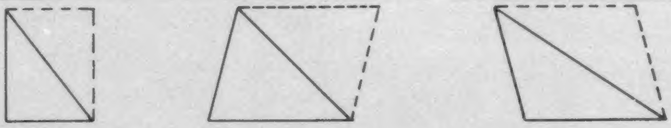
With repetitions of the experiment using various dimensions of the rectangular region and parallelogram region, students should discover the formula $A = bh$ for parallelogram regions. You may need to help them make the transfer from $A = lw$ to $A = bh$.

Measurement **Teacher Card 8b**

Related Activity

Prepare similar task cards or activity for discovering the area of triangular regions. Once students know the formula for area of a parallelogram region, $A = bh$, discovery of the formula of a triangular region is one step away.

Given any triangular region, its area will be half of the corresponding parallelogram region.



Students should investigate enough examples to be convinced that $A = 1/2\ bh$ is the formula for the area of any triangular region. You may consider the cases shown above — right triangle, acute triangle and obtuse triangle.

Variation This activity could be modified for use with geoboards instead of grid paper.

Measurement **Student Card 9a**

Objectives Select the appropriate type of measurement needed for a given physical situation, e.g., length, area, volume

Select the appropriate formula(s) for finding the measurement for a given situation, e.g., area, perimeter, circumference, volume

Materials Measuring stick

Activity

1. Study the chart below. There is a sample problem and information needed to solve the problem.
2. Make up three more problems — one where you need to find length or perimeter, one where you need to find area and another where you need to find volume or capacity. Write the problems in the chart on Student Card 9b and then complete the chart for each one.
3. Choose one of the problems and solve it by measuring and using the appropriate formula.

MEASUREMENT PROBLEMS

Problem	I will need to find	Number of Dimensions	Possible Labels	Length	Width	Thickness or Height
To cover my desk with contact paper, how much do I need?	Area	2	cm^2 or sq. in.			not used

Measurement **Student Card 9b**

MEASUREMENT PROBLEMS

Problem	I will need to find	Number of Dimensions	Possible Labels	Length	Width	Thickness or Height

Measurement **Teacher Card 9a**

Objectives Select the appropriate type of measurement needed for a given physical situation, e.g., length, area, volume

Select the appropriate formula(s) for finding the measurement for a given situation, e.g., length, area, perimeter, circumference, volume

Materials Measuring stick

Processes Problem finding, problem focusing, measuring, intrepreting

Notes This activity may be one of the more difficult for students to complete. Finding problems and then deciding on appropriate interpretation entails a good deal of reasoning.

Possible problem statements include the following.

- For length or perimeter — Find the length of molding needed to go all the way around the classroom (perimeter).
  What is the distance from my classroom door to the principal's office?
- For area — If we were to carpet this classroom, how many square yards of carpet would we need?
  What is the size of the chalkboard? (This could be area or perimeter depending on interpretation)

Measurement **Teacher Card 9b**

- For volume or capacity — How much water do I need to fill the sink (in the classroom or restroom)?
  How many balloons would it take to fill this classroom?

Some students will create a variety of problems. Others may need help to think of even one problem situation. You may want to have students work together on this task or later share problems with each other.

It is important that students experience the unfamiliar (to many of them) situation of finding problems and then finding methods for solving them. This activity also helps show the need for thoughtful interpretation and decisions on **what to solve for** before any solving can begin.

Students may find that some of their problems are not stated clearly enough to begin to solve. They may need to reformulate their problem or question. Others may find that their problem is too difficult to solve. Since each student can choose from three problems, there should be a do-able one in the set. You may need to guide students who are having trouble deciding what to do.

Measurement **Student Card 10a**

Objectives Derive formulas for finding perimeter of simple closed curves

Find the mean of a set of numbers

Materials Five tin cans of various diameters, a metric measuring tape and a calculator

Activity

1. Measure the circumference (distance around) and the diameter (distance across the top) of five tin cans. Measure to the nearest tenth of a centimeter (or millimeter) and record this data in the Chart on Student Card 10b.
2. Use the calculator to divide the circumference by the diameter for each tin can and record this quotient in the chart. Express your quotient with four decimal places.
3. Use the calculator to find the mean (average) of the quotients you have calculated. Express this answer with four decimal places. ________
4. What relationship did you discover between the circumference and the diameter of your tin cans?
5. Check to see a friend's findings.

Measurement **Student Card 10b**

FINDING CIRCUMFERENCE

Tin Can	Circumference	Diameter	C ÷ D
1.			
2.			
3.			
4			
5.			

6. If you know the diameter of a tin can, how would you find its circumference (without measuring)?
7. The top of the tin can is a circle. Write a formula for the circumference of a circle in terms of its diameter. C =

Measurement **Teacher Card 10a**

Objectives Derive formulas for finding perimeter of simple closed curves

Find the mean of a set of numbers

Materials Tin cans of various diameters, metric measuring tapes, calculators and metric rulers (optional)

Processes Measuring, computing with a calculator, comparing, looking back, averaging, deriving, generalizing

Modifications for Special Students

- Visually impaired students will need their modified tape measures. See Teacher Card 2a. They may also need a partner to help read the calculator display.

Notes Ask parents to help save old tin cans for this activity. With about two weeks notice, you will have plenty of cans.

Consider having pairs of students work together. Holding the tape measure around and across the top of the can is easier with two. You might also want to provide rulers for measuring across the top of the can since a ruler is less likely to bend into the can and is easier to manage.

Measurement **Teacher Card 10b**

For C ÷ d, students should get answers near the value of π or approximately 3.1416. While the individual answers for C ÷ d will not be this accurate, the average should be close to 3.1416.

Lead students to realize that given the diameter of a can, they can find the circumference by multiplying the diameter by 3.1416 (π), i.e.,

$$C = 3.1416 \cdot d \quad \text{or} \quad C = \pi d$$

Some students may also need help realizing that their measuring circumference and diameter of a tin can is essentially the same as measuring the circumference and diameter of a physical representation of a circle. Holding the can up and looking at its top will probably be enough for many students. You might prepare ahead of time a can with the bottom and top cut out for those students who are harder to convince.

Measurement **Teacher Card 11a**

Objectives Select the appropriate type of measurement needed for a given problem situation, e.g., length, area, volume

Select the appropriate formula(s) for finding the measurement for a given situation, e.g., area, perimeter, circumference, volume

Use formulas to solve geometric problems involving perimeter, area and volume

Materials Grid paper, catalogues, newspapers and magazines

Processes Modeling, interpreting, scaling, decision making, approximating, using resources

Teacher-initiated Activity—Design a Room

Have the students decide what they would like their dream room to be like. They should draw out their rooms on grid paper and label the dimensions of the room and items in the room. Use catalogues to help plan the use of wallpaper, paint, curtains, carpeting, etc.

Have students make lists of those materials that they would need to fix up their rooms. Have them list sizes and quantities needed and how much these materials would cost. Have them submit a cost estimate to decorate the room from floor to ceiling.

Measurement **Teacher Card 11b**

Notes Using an activity on decorating the students' rooms as a motivator, your students will be involved in a realistic problem solving situation. They will begin to understand as you point out the physical situation — their room — and the mathematical model — their design on grid paper.

They will need to take measurements at home before beginning to reduce the dimensions to grid paper. The idea of scale will become important, e.g., one centimeter will represent one decimeter (or meter or foot).

Estimation of cost may become a factor, particularly for the thrifty students. You may let students pretend that money is not a factor or you may limit them to a specified sum. Constraints such as limited funds give the problem a different character.

Students will need to select appropriate types of measurements needed and then select the appropriate formulas and apply them. They will view the mathematics they are learning as useful for solving problems that become meaningful to them.

Follow-up-Activity

Have an interior designer come to speak to the class. You may have a parent help locate such a person.

Ratios, Proportions and the Great Outdoors
Relations and Functions

This unit focuses on ratio and proportion. For the most part, students will participate in exploration activities outside the classroom. Many of the activities, although begun outside, can be finished inside. The teacher may wish to plan access to a chalkboard which can be taken outside. While the majority of objectives in these activities are from the Relations and Functions Strand, others are included. The objectives are listed below.

Objective		**Remarks**
Relations and Functions		
1	Classify elements of a set according to specified properties	Student Card 1
2	(Partial) Demonstrate correspondences (a) one-to-one, (b) one-to-many, (c) many-to-one and (d) many-to-many	Student Card 1
3	(Partial) Apply equivalence relations to elements such as to fractions, ratios, percents and measures of geometric figures, e.g., set up a proportion	Student Cards 2, 3 and 4 Teacher Card 6 Parent Card 1
4	Find some pairs of elements when a relation is given, e.g., given the relation **square the number** some pairs of elements are (2, 4), (3, 9), (5,25), (10,100)	Teacher Card 5
5	Find the missing element of a pair when one member of the pair and the relation are given, e.g., given 4 and the relation **multiply by 3,** then the missing element is 12.	Teacher Card 5
8	Use the addition and multiplication properties of equality to solve one-variable open sentences, e.g., if $A = l \cdot w$, then $l = A/w$ or solve for x in the proportion $8/6 = 20/x$	Student Card 4 Teacher Card 6
10	(Partial) Use a graph in coordinate plane to represent (a) ordered pairs (b) the solution set of an equation (c) the solution set of an inequality	Student Card 3 Teacher Card 5
11	(Partial) Interpret a graph (a) on a number line and (b) in a coordinate plane	Teacher Card 5
Measurement		
13	(Partial) Determine a length or distance using (a) nonstandard units and (b) standard units	Student Cards 2, 3 and 4
Sets, Numbers and Numeration		
12	Change one number representation to another representation, e.g., given 40%, change to 0.4; given 2/5 generate some equivalent fractions 4/10, 6/15, 12/30, 40/100; change 1/3 to $0.\bar{3}$; and $^-(^-2) = 2$	Student Card 2
13	Apply an appropriate number representation to a particular situation, e.g., the discount of a $400 item with 25% off can be found by 1/4 × 400 or 0.25 × 400	Student Card 2

Geometry

14 Solve simple geometric problems using properties of similar figures, e.g., indirect measurement

Student Card 4

The following processes are among those used in carrying out the activities.

- Observing
- Describing
- Classifying
- Experimenting
- Recording
- Estimating
- Looking back
- Relating
- Comparing
- Predicting
- Measuring
- Cooperating with others
- Computing
- Applying
- Generalizing
- Interpreting

Relations and Functions **Student Card 1a**

Objectives Classify elements of a set according to specific properties

Demonstrate correspondences one-to-one and many-to-one

Materials Patch of ground, a meter stick, paper and pencil

Activity

1. Mark off a square meter section of ground with anything available (rocks or sticks make good boundaries). Observe the patch and make a list of all the items you find there. Try to find as many as possible.
2. Using your list, answer these questions.

   Do some of the items have common properties? If so, what are the items and the properties.

   How are the items or groups of items different from one another?
3. Fill in the chart on Student Card 1b to show the relationship between items with like properties by using the rule "is a type of". An example is done for you.

   Read the chart this way. "Grasshopper is a type of animal."

Relations and Functions **Student Card 1b**

CLASSIFYING

1. grasshopper	animal	1. grass	vegetable	1. rock	mineral
2. ________		2. ________		2. ________	
3. ________		3. ________		3. ________	
4. ________		4. ________		4. ________	

4. Try to separate your list of items by using your own rule.
   What is your rule?
5. Make a chart like the one above using your own rule. Give the chart to a friend and see if he or she can guess what rule you used.

Relations and Functions **Teacher Card 1a**

Objectives Classify elements of a set according to specified properties

Demonstrate correspondences one-to-one and many-to-one

Materials Patches of ground, meter sticks, portable chalkboard (optional)

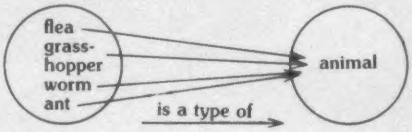
Processes Observing, describing, classifying

Modifications for Special Students

- Visually impaired students may need help **seeing** objects in a patch of ground. Pair this student with a sighted student.

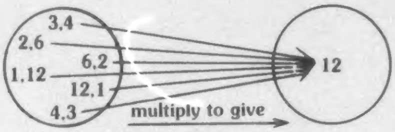
Notes Ask students to make a list of objects found in their patches of ground. They should classify these objects according to specified properties.

Point out that the lists down in Student Card 1b show many-to-one correspondence. Draw a diagram like the following to show many-to-one correspondence. This may be done outside with a portable chalkboard or the activity may be continued in the classroom.



Relations and Functions **Teacher Card 1b**

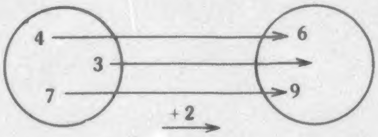
Give an example using numbers.



Ask students to make a diagram showing many-to-one correspondences with the items on their lists.

Ask students if anyone had a category with only one item in it. Point out that this is a one-to-one correspondence. If no one had any one-to-one correspondence, point out some like noses to faces.

Show a one-to-one correspondence by diagram using numbers.



Have students suggest other examples both numerical and nonnumerical.

Relations and Functions **Student Card 2a**

Objective Determine a length or distance using standard units

Apply an appropriate number representation to a particular situation

Change one number representation to another representation

Apply equivalence relations to elements such as fractions and ratios

Materials A few balls of different sizes and types, a meter stick, a tape measure about 2 meters long, tape and a calculator

Activity

1. For this activity you will need a partner. Tape the 2-meter tape measure to the wall so that zero is at the floor and the other end goes toward the ceiling. Drop one of the balls from a height of 50 centimeters. Have your partner observe the ball to see how high it bounces. This may take some practice. Exchange jobs with your partner and try this again.
2. Continue to drop the ball from different heights and fill in the chart on Student Card 2b. You may want to take more than one reading each time for accuracy.

Relations and Functions **Student Card 2b**

RECORD OF BOUNCING BALLS

Height of Bounce	________	________	________	________	________
Height of Drop	50cm	100cm	150cm	200cm	

3. If you were careful the ratios you wrote should be equivalent. Predict the height of a bounce if you dropped the ball from 250 centimeters. Try it and see how close you were. Write it in the chart.
4. Use a calculator to divide the height of the bounce by the height of the drop for each of the trials from the chart. This will give decimal numerals which are the decimal representations of the ratios in the chart.
5. What do you notice about these decimal numerals?
6. What would the height of the bounce be if you could drop the ball from one centimeter.
7. Repeat the experiment using balls of different sizes and types.

Relations and Functions **Teacher Card 2a**

Objectives Determine a length or distance using standard units

Apply an appropriate number representation to a particular situation

Change one number representation to another representation

Apply equivalence relations to elements such as fractions and ratios

Materials Balls of different sizes and types, meter sticks, tape measures about two meters long, tape and a calculator for each pair of students
If you don't have a tape measure, the students can make them. Use a standard meter stick and adding machine tape. Have the students mark off centimeters on it.

Processes Experimenting, measuring, recording, estimating, looking back, comparing, predicting

Modifications for Special Students

- Visually impaired students may not be able to observe how high the ball bounces. These students may be the ones to always drop the ball.

Relations and Functions **Teacher Card 2b**

Notes Have students work in pairs for this activity. Encourage each of the partners to take turns dropping the ball and observing.

For a particular ball, the ratio of height of bounce to height of drop will be constant, i.e., the ratios will be equal. A certain amount of error is introduced in reading the height of the bounce. Discuss this with your students as the ratios they find may be **exactly** the same.

Use of the calculator will help students become aware of the relationship between fractions and decimals. It will also be more obvious that the ratios are close to each other. For example, if a student found

$$\frac{24}{50} \;,\; \frac{49}{100} \;,\; \frac{76}{150} \;,\; \frac{100}{200}$$

and the decimal equivalents rounded to hundredths,

$$.48 \;,\; .49 \;,\; .51 \;,\; .50$$

the fact that these numbers are nearly equal becomes more evident. If the students are still unsure, have them round to the nearest tenth and all the ratios will be the **same,** *i.e.,* .5 .

Relations and Functions **Student Card 3a**

Objectives Determine a length or distance using standard units

Apply equivalence relations to elements such as ratios

Use a graph in a coordinate plane to represent ordered pairs

Materials A metric tape measure or meter stick and graph paper

This activity must be done on a sunny day.

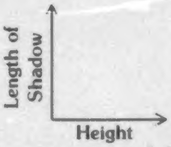
Activity 1. You and your partner will measure the heights and shadows of each other. Take the shadow measurements at nearly the same time. Record them in the chart below. Collect data from two other pairs of students so that you have information for a total of six students.

RECORD OF SHADOW MEASUREMENTS

Student	Height	Length of Shadow
1.		
2.		
3.		
4.		
5.		
6.		

Relations and Functions **Student Card 3b**

2. Look at the example and using the chart, make a graph on your graph paper.
   Your graph will contain six points.



3. Why is it important to take shadow measurements at the same time of day?
4. How would the graph be different if the shadows were measured at different times?
5. Predict the length of the shadow of a six-foot tall person for the same time of day you took your readings.

Relations and Functions — Teacher Card 3a

Objectives Determine a length of distances using standard units

Apply equivalence relations to elements such as ratios

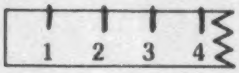
Use a graph in a coordinate plane to represent ordered pairs

Materials Metric tape measures or meter sticks, graph paper and chalk (optional)

Processes Measuring, recording, graphing, comparing, cooperating with others, predicting

Modifications for Special Students

- Visually impaired students may need adaptation of the meter stick. You can make marks they can **feel** for centimeter units. Along each centimeter mark, squirt a thin line of glue and sprinkle with sand.



A parent may be called upon to help make such materials.

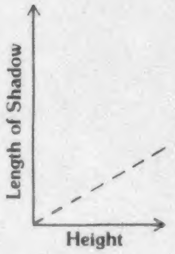
- Orthopedically impaired students may need extra assistance standing for this activity or height while sitting may be used.

Relations and Functions — Teacher Card 3b

This activity must be done on a sunny day.

Notes Students will need to work in pairs. Have them stand outside on a hard, flat surface. Have them measure their heights and lengths of their shadows. You may want to give them chalk to mark their shadows.

Ask if all the shadows are the same length. Ask students if they can find any relationships between the heights and the shadow lengths. If they cannot, you may need to ask questions such as, "Are all the shadows longer than their corresponding person? Shorter?"



Points on students' graphs of heights and lengths of shadows should lie on a straight line. Small errors in measuring and time factors could alter this slightly. Point out these types of errors to the students. Discuss relationships between equal ratios and points on the students graphs, i.e., the graph of an equivalence relation is a straight line.

Send home Parent Card 1.

Extensions Write each ratio as a fraction: height/length of shadow. Simplify and have students compare their ratio with those of other students. Calculators can be used to find decimal equivalents.

Measurement of other objects could be suggested. Measuring at different times of the day might be pursued possibly as a weekend project, e.g., heights of members of a family and corresponding shadow lengths.

Relations and Functions **Student Card 4a**

Objectives Determine a length or distance using standard units.

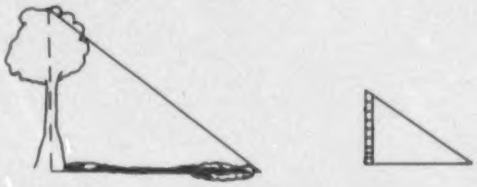
Solve simple geometric problems by using properties of similar figures, *e.g.*, indirect measurement

Apply equivalence relations to elements such as measures of geometric figures

Use the addition and multiplication properties of equality to solve for one-variable open sentences, e.g., solve a proportion

Materials Measuring tape, a meter stick, a sunny day and a tall tree

Activity 1. Measure the shadow of a tall tree. Hold the meter stick so that it is parallel to the tree trunk and measure its shadow.



Similar triangles are formed by the objects, their shadows and the rays of the sun as shown above.

Relations and Functions **Student Card 4b**

2. Is the shadow of the meter stick longer or shorter than the stick?

   Is the shadow of the tree longer or shorter than the tree?

3. Write a ratio comparing the height of the meter stick to the length of its shadow.

$$\frac{\text{stick height}}{\text{stick shadow}}$$

Write the ratio comparing the height of the tree (an unknown) to the shadow of the tree.

$$\frac{\text{tree height}}{\text{tree shadow}}$$

Since two similar triangles are formed as shown in the diagram, these two ratios are equal. Write the two ratios in the form of a proportion.

$$\frac{\text{tree height}}{\text{tree shadow}} = \frac{\text{stick height}}{\text{stick shadow}}$$

4. Solve the proportion to find the tree height.

Relations and Functions **Teacher Card 4a**

Objectives Determine a length or distance using standard units

Solve simple geometric problems by using properties of similar figures, e.g., indirect measurements

Apply equivalence relations to elements such as measures of geometric figures

Use the addition and multiplication properties of equality to solve for one-variable open sentences, e.g., solve a proportion

Materials A sunny day, metric measuring tapes, meter sticks, a tall tree

Processes Measuring, recording, comparing, computing

Modifications for Special Students

- Visually impaired students may need the adapted meter stick described in Teacher Card 3a.

Notes You may want students to work in pairs so that one can hold the meter stick and the other mark or measure its shadow.

Students may need some help to solve the proportion they get. For example, a student may have

$$\frac{\text{tree height}}{524 \text{ cm}} = \frac{100 \text{ cm}}{65 \text{ cm}}$$

Relations and Functions **Teacher Card 4b**

One method of solving the proportion is to see that the proportion can be written

$$(\text{tree height}) \times (65 \text{ cm}) = (524 \text{ cm}) \times (100 \text{ cm})$$

or

$$65 \times \text{tree height} = 52{,}400 \text{ cm}$$

You can then lead the students to a solution method by asking "What number times 65 equals 52,400?" This is essentially a division problem.

$$52{,}400 \div 65$$

For this example, the height of the tree would be approximately 806 centimeters or just over 8 meters.

Extensions Possible extensions include measuring other tall trees using this method of indirect measurement.
Flag poles, the students themselves, and so on can all be measured indirectly in such a manner.

Relations and Functions **Teacher Card 5a**

Objectives Find some pairs of elements when a relation is given, e.g., given the relations **square the number** some pairs of elements are (2, 4), (3, 9), (5, 25), (10, 100)

Find the missing element of a pair when one member of the pair and the relation are given, e.g., given 4 and the relation **multiply by 3,** then the missing element is 12

Use a graph in a coordinate plane to represent (a) ordered pairs

Interpret a graph in a coordinate plane

Materials Graph Paper

Processes Relating, generalizing, applying, graphing, interpreting

Teacher-led Activity

1. Ask students what would be some of the possible ratios of heights to shadows if the shadow at a certain time of day is twice as long as the object.
2. Ask students to list ratios that would occur if the shadow is half as long as the object.

Relations and Functions **Teacher Card 5b**

3. Ask the student to list ratios that would occur if a two-foot tall object had a five-foot tall shadow.
4. Have students fill in the chart of height to lengths of shadows. Use the following information A six-foot tall man has a four-foot tall shadow.

Object	Height	Shadow	(Missing Elements)
man	6	4	
post		2	(3)
pole	12		(8)
tree		10	(15)

5. Have students use the information in above chart to graph the ordered pairs of the relation.
6. Lead a discussion on how to interpret the graph and possibly use it to make predictions.

Relations and Functions **Teacher Card 6a**

Objectives Apply equivalence relations to elements such as fractions and ratios, e.g., set up a proportion.

Use addition and multiplication properties of equality to solve one-variable open sentences, e.g., solve for x in the proportion $8/6 = 20/x$.

Materials A five- or ten-speed bicycle, masking tape

Processes Observing, comparing, relating, predicting

Teacher-led Activity

1. Turn the bicycle upside down. Have the students observe the placement of the chain. Have them count the number of teeth in each of the gears which are surrounded by the chain. Write the numbers on the chalkboard as the ratio of the number of front gear teeth to the number of back gear teeth.

$$\frac{\text{number of front gear teeth}}{\text{number of back gear teeth}}$$

Relations and Functions **Teacher Card 6b**

2. Mark the back wheel with a piece of masking tape so that the number of times it turns can be determined. Have a student turn the pedals the same number of times as the number of teeth in the front gear. Have the other students count how many times the back wheel goes around. This should be the same as the number of back teeth. Write the ratio of the number of times the pedal went around to the number of times the wheel went around. Point out that the ratios are equal and form a proportion.
3. Ask the students what would happen if the pedal were turned twice as many times or half as many times? Have them verify their estimate by trying it.
4. Do this again with other gear combinations. Count the teeth, set up proportions and check the gear ratio by turning the pedals and observing the number of times the back wheel goes around.

Relations and Functions **Parent Card 1**

Objective Apply relations (or pairings of numbers) such as fractions, ratios and geometric figures

Activity

1. When shopping point out relations or number-pairs that are used. For example, 1 pencil cost 10 cents, 2 pencils cost 20 cents and 3 pencils cost 30 cents.
2. At the gas station, observe the numbers on the pump. If you pump your own gas, stop the pump at 1, 2 and 3 gallon readings so that your child can make a chart showing the relationship between the amount of gas you buy and the cost.

COSTS OF GASOLINE

gallons	cost
1	
2	
3	
4	
5	

Evaluating Mathematics Learning

MATHEMATICS LEARNING

Evaluating Mathematics Learning

In several other sections of this guide there are suggestions for evaluating mathematics learning. This section will provide a point of view on evaluation, develop a framework for discussing evaluation and relate this material to other sections of the guide. A basic assumption is that evaluating mathematics learning requires a certain expertise in mathematics teaching. This is not a task that can be left to the outside specialist. The evaluation of mathematics learning must follow from the objectives and strategies for mathematics instruction. The process must not be allowed to reverse itself, letting the principles of evaluation dictate the objectives of mathematics instruction.

What Is Evaluation?

Evaluation is an integral part of a successful mathematics program. It includes assessing mathematics learning, selecting and improving instructional materials, observing teaching techniques and judging the quality of mathematics programs. However, evaluating is certainly more than testing.

To be sure, valid mathematics tests are important for some of the evaluation of mathematics learning and programs. But, they are instruments of the evaluation, always with certain limitations, and the data from tests are only one element of the evaluation.

To evaluate is to tell a story. Evaluation must weave together all available information critical to the story (for instance, observations of the students, observations of the testing, descriptions of the program, the teaching or the students' backgrounds). The evaluation process ultimately rests on judgments. The human mind, then, becomes an evaluation instrument. Judgments are made concerning the selection of tests, the interpretation of the data and, of course, the decisions made.

The evaluation process, therefore, is a sequence of information-gathering, interpretation and decision-making. In determining a student's grade on a unit, for instance, the teacher gathers information from tests, homework, classroom interaction and observation. The teacher makes judgments on the relative importance of the different kinds of information and finally arrives at a decision concerning the grade. The teacher is guided by concerns for quality, fairness, validity and reliability of the information and consistency, at the same time the sequence of information gathering, interpretation and decision making has been followed. Hence, the letter grade is only a small part of the total picture.

Similarly, in evaluating a mathematics program, someone gathers information (objectives, test data, program descriptions), provides an analysis of the information and presents the result in a manner that reflects an interpretation of "what it's all about."

Why Evaluate?

Information about mathematics learning is usually gathered because a decision of some sort has to be made.

- Has the student learned a specific objective?
- Have objectives been met satisfactorily?
- Should a particular textbook be used?
- Has a program worked?

Evaluation of mathematics learning, therefore, is concerned with the whole range of decisions that must be made when organizing and implementing a mathematics program.

Evaluation is an integral part of the instructional process. Assessing mathematics performance helps students set personal goals, provides them with indications of their progress and may even motivate them. Evaluation is also a source of reinforcement and feedback. It may even provide a sense of stability and organization for mathematics as perceived by the student.

Additionally, evaluation serves to provide information to the various groups to whom the schools should be accountable. This includes parents, those who determine school policies and funding and the public. Mathematics evaluation provides not only the opportunity to describe how programs are doing—through test scores and the like—but also, if thoroughly reported, the chance to discuss the nature of the program. Unfortunately, reporting often simply gives scores with no discussion of the program and misrepresents the mathematics program by implying that the content of the test exemplifies the curriculum.

Another reason that mathematics should be evaluated is that it is often imperative, prescribed, in some cases, by mandates from outside the school system.

Given that several answers to the question "why evaluate?" exist, it is apparent that mathematics evaluation will be done. Therefore, it should be done well. To do it well requires interest, involvement and the direction of mathematics teachers at every step—from the selection of objectives to using and reporting the data.

Questions to Consider When Evaluating the Student Materials of a Mathematics Program

You Should Also Evaluate the Teacher Materials. Consider These Questions.

- How much in-service work on the part of prospective teachers will be required to implement the program?
- Who provides the in-service and at what additional cost?
- Are supplementary resources suggested?
- How much development of materials will be required to adapt the program to meet the needs of the local program?
- Do the program objectives match with the local mathematics program objectives?
- Are manipulative materials suggested for program implementation? (See the *Thirty-fourth Yearbook of NCTM, Instructional Aids in Mathematics*, for criteria for evaluating a manipulative device.)
- Are solutions included where applicable?
- Are varied instructional strategies suggested and described for classroom implementation?
- Does the program lend itself to a variety of instructional strategies? For instance, is the program sensitive to problem solving as a small group activity? Is opportunity built into the program for the students to talk mathematics to one another?
- Is the program correlated with other content areas?
- Does the program include both formative and summative evaluation?

Do the Student Materials

- include topics outlined in the local mathematics curriculum guide?
- reflect the needs of the local mathematics program as identified by the Statewide Criterion-Referenced Test?
- exemplify consistency, accuracy and precision in vocabulary and symbols?
- include sufficient practice material?
- incorporate real-world applications as well as concept development?
- encompass consumer and career awareness?
- embody a consideration of the use of hand-held calculators and computers?
- attend to the teaching of the metric system of measurement (with no conversion between English and metric units)?
- include an early treatment of decimals?
- comprise motivational devices?
- contain sufficient materials for reinforcement, diagnosis and evaluation (including self-evaluation)?
- review previous-level concepts throughout the program?
- exhibit readability level appropriate to students for whom intended?
- provide a variety of topics and differentiated activities endemic to age and interests and relative to different achievement levels?
- make appropriate use of charts, tables and graphs?

Instructional Resources

Instructional Resources to Support the Middle Grades Mathematics Program

References

Instructional Resources to Support the Middle Grades Mathematics Program

Educational media programs in Georgia public schools focus not only on the provision of instructional resources in all formats to support the curriculum but also on the utilization of those resources in supporting teaching strategies and learning activities to meet student needs effectively. A combination of resources including print and nonprint materials and equipment essential for their utilization or production along with programs, services and additional resources available through state, community and other educational agencies are necessary for effective support of instructional programs.

Innovative teachers, media specialists, administrators, curriculum specialists, students, board members and representatives of the community are cooperatively evolving a media concept that supports the instructional program and facilitates access to information in all formats and provision of services in production of locally designed, curriculum related learning materials. Media specialists should serve on curriculum committees and integrate their professional skills in a cooperative effort to develop effective curriculum programs. In addition, efficient utilization of appropriate materials fostering student growth in listening, viewing, reading and inquiry skills is being increased by these populations. Georgia Board of Education Instructional Media and Equipment Policy requires that local media committees composed of the groups mentioned above be involved in selecting materials and establishing procedures for using them effectively. Mathematics teachers should express to their principal and media specialist their interest in being involved in or providing input to this planning process.

Ensuring access of teachers and students to information at the time of need and preventing unnecessary duplication of resources will be accomplished when information about and location of resources that support the mathematics program in a middle school are available through the school's media center. Through involvement in or input to such activities as policy and procedure development, curriculum design and materials evaluation and selection, mathematics teachers have an opportunity and a responsibility in the development of improved media services supporting the instructional program.

A community resources file, developed cooperatively by media and instructional staff, provides valuable additional information about local people, places, activities and unique resources to enhance the mathematics program. In some school systems, a resource service designed to augment the building media program is also provided at system level for all schools.

Numerous sources of information about resources exist. Some are commercially prepared, others are provided by the Georgia Department of Education while others exist in the local school. Media personnel in each building media center can assist teachers in using the following suggested sources.

Resources from the U.S. Department of Education

Many professionally prepared commercially published reviewing sources which are available in school media centers, system media collections and public and academic libraries are listed in *Aids to Media Selection for Students and Teachers*, available from U.S. Department of Health, Education and Welfare, Office of Education, Bureau of Elementary and Secondary Education, Office of Libraries and Learning Resources, Washington, D.C.

Resources from the Georgia Department of Education

The Department of Education provides resources and services which are available through school media centers.

The following materials distributed by the Division of Educational Media Services are available through school media centers and/or the System Media Contact Person (Educational Media Services Division, Instructional Resources Unit, Georgia Department of Education, Twin Towers East, Atlanta, Georgia 30334).

Georgia Tapes for Teaching: Catalog of Classroom Teaching Tapes for Georgia Schools (and supplements). Arranged by subjects, this catalog lists the titles of audio tapes which will be duplicated on request. Recommended listening audiences are indicated. A school registration is required. The requesting media center must provide the blank reel-to-reel or cassette tape on which the recording is made. Return postage is provided by the Georgia Department of Education.

Catalog of Classroom Teaching Films for Georgia Schools (and supplements). The annotated list of 16mm films is arranged by titles but indexed by subjects. Recommended viewing audiences are indicated. Registration (annual beginning in September or semi-annual beginning in January) requires a minimal fee. Each registration provides a specified weekly film quota, but multiple registrations are accepted. Many films are broadcast over the Georgia Educational Television Network and some may be duplicated on videotapes for later use. Information about this service and the broadcast schedule are provided annually to the System Media Contact Person.

Instructional Television Schedule. Copies of the schedule with series descriptions and broadcast times are available on request from your System Media Contact Person, who also coordinates orders for needed teacher manuals. Descriptions of telecourse series and programs in related fields should be examined for potential programs to support the mathematics curriculum. Although recommended viewing audiences are indicated, the schedule and/or teacher manuals should also be examined for potential use of a program or series to introduce, develop, or reinforce mathematical concepts. Upcoming broadcast specials are announced in MEDIA MEMO which is provided monthly during the school year by the Department of Education.

A bibliography, "Selected Sources of Information on Educational Media", is also available from this Division: Media Field Services, Division of Educational Media Services, Twin Towers East, Atlanta, Georgia 30334.

Additional sources of information provided by the Georgia Department of Education are

Educational Information Center (EIC), Georgia Department of Education, Twin Towers East, Atlanta, Georgia 30334.

Research service is provided to Georgia public school administrators and their central office staff. Computer and manual searches of Educational Resources Information Center (ERIC) data base which includes over 325,000 references to education documents related to exemplary projects and model teaching strategies can be requested by the media staff through the System Media Contact Person.

Readers Service, Public Library Services Division, Georgia Department of Education, Atlanta, Georgia 30334.

"Selected List of Books for Teachers" (and supplements) and "Periodical List" (and supplements) identifying titles in the Georgia Public Library Information Network (GLIN), another reference and bibliographic service, provides access to publications in the collections of approximately 150 participating public, special and academic libraries. Requests for these services and resources should be made through the local public libraries by the school media staff.

Other Sources of Information/Ideas

Reviews and bibliographies of recommended resources and innovative program descriptions for mathematics are published regularly in journals and periodicals. The following tools for selection of titles are recommended.

Arithmetic Teacher, National Council of Teachers of Mathematics, monthly September-April

Computing Teachers, The Oregon Council for Computer Education, six issues per year

Creative Computing, Creative Computing Association, bimonthly

Mathematics Teacher, National Council of Teachers of Mathematics, monthly September-May

Science Books & Films, American Association for the Advancement of Science, quarterly

School Science and Mathematics, School Science and Mathematics Association, monthly October-May

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Appendices

Appendix A
Organization for the Essentials of Education

(a statement written and endorsed by professional educational organizations)

Society must reaffirm the value of a balanced education.

Leaders of several professional organizations reached this conclusion in 1978. They circulated a statement on the essentials of education among a number of professional associations whose governing boards endorsed the statement and urged that it be called to the immediate attention of the entire education community, of policy makers and of the public at large.

The statement that follows embodies the collective concern of the endorsing associations. It expresses their call for a renewed commitment to a more complete and more fulfilling education for all.

The associations invite the concurrence, support and participation of everyone interested in education.

The Essentials of Education

Public concern about basic knowledge and the basic skills in education is valid. Society should continually seek out, define and then provide for every person those elements of education that are essential to a productive and meaningful life.

The basic elements of knowledge and skill are only a part of the essentials of education. In an era dominated by cries for going "back to the basics," for "minimal competencies," and for "survival skills," society should reject simplistic solutions and declare a commitment to the essentials of education.

A definition of the essentials of education should avoid three easy tendencies: to limit the essentials to "the three R's" in a society that is highly technological and complex; to define the essentials by what is tested at a time when tests are severely limited in what they can measure; and to reduce the essentials to a few "skills" when it is obvious that people use a combination of skills, knowledge and feelings to come to terms with their world. By rejecting these simplistic tendencies, educators will avoid concentration on training in a few skills at the expense of preparing students for the changing world in which they must live.

Educators should resist pressures to concentrate solely upon easy-to-teach, easy-to-test bits of knowledge, and must go beyond short-term objectives of training for jobs or producing citizens who can perform routine tasks but cannot apply their knowledge or skills, cannot reason about their society, and cannot make informed judgments.

What, Then are the Essentials of Education?

Educators agree that the overarching goal of education is to develop informed, thinking citizens capable of participating in both domestic and world affairs. The development of such citizens depends not only upon education for citizenship, but also upon other essentials of education shared by all subjects.

The interdependence of skills and content is the central concept of the essentials of education. Skills and abilities do not grow in isolation from content. In all subjects, students develop skills in using language and other symbol systems; they develop the ability to reason; they undergo experiences that lead to emotional and social maturity. Students master these skills and abilities through observing, listening, reading, talking, and writing **about** science, mathematics, history and the social sciences, the arts and other aspects of our intellectual, social and cultural heritage. As they learn about their world and its heritage they necessarily deepen their skills in language and reasoning and acquire the basis for emotional, aesthetic and social growth. They also become aware of the world around them and develop an understanding and appreciation of the interdependence of the many facets of that world.

More specifically, the essentials of education include the ability to use language, to think, and to communicate effectively; to use mathematical knowledge and methods to solve problems; to reason logically; to use abstractions and symbols with power and ease; to apply and to understand scientific knowledge and methods; to make use of technology and to understand its limitations; to express oneself through the arts and to understand the artistic expressions of others; to understand other languages and cultures; to understand spatial relationships; to apply knowledge about health, nutrition, and physical activity; to acquire the capacity to meet unexpected challenges; to make informed value judgments; to recognize and to use one's full learning potential; and to prepare to go on learning for a lifetime.

Such a definition calls for a realization that all disciplines must join together and acknowledge their interdependence. Determining the essentials of education is a continuing process, far more demanding and significant than listing isolated skills assumed to be basic. Putting the essentials of education into practice requires instructional programs based on this new sense of interdependence.

Educators must also join with many segments of society to specify the essentials of education more fully. Among these segments are legislators, school boards, parents, students, workers' organizations, businesses, publishers, and other groups and individuals with an interest in education. All must now participate in a coordinated effort on behalf of society to confront this task. **Everyone** has a stake in the essentials of education.

Professional Associations Endorsing This Statement

American Alliance for Health, Physical Education, Recreation and Dance
1201 16th Street, N.W., Washington, D.C. 20036 (202) 833-5553

American Council on the Teaching of Foreign Languages
2 Park Avenue, Room 1814, New York, New York 10016 (212) 689-8021

Association for Supervision and Curriculum Development
225 North Washington Street, Alexandria, Virginia 22314 (703) 549-9110

International Reading Association
800 Barksdale Road, P.O. Box 8139, Newark, Delaware 19711 (302) 731-1600

Music Educators National Conference
1902 Association Drive, Reston, Virginia 22091 (703) 860-4000

National Art Education Association
1916 Association Drive, Reston, Virginia 22091 (703) 860-8000

National Association of Elementary School Principals
1801 N. Moore Street, Arlington, Virginia 22209 (703) 528-6000

National Council for the Social Studies
3615 Wisconsin Avenue, N.W., Washington, D.C. 20016 (202) 966-7840

National Council of Teachers of English
1111 Kenyon Road, Urbana, Illinois 61801 (217) 328-3870

National Council of Teachers of Mathematics
1906 Association Drive, Reston, Virginia 22091 (703) 620-9840

National Science Teachers Association
1742 Connecticut Avenue, N.W., Washington, D.C. 20009 (202) 265-4150

Speech Communication Association
5205 Leesburg Pike, Falls Church, Virginia 22041 (703) 379-1888

Appendix B
Position Paper on Basic Mathematical Skills of the National Council of Supervisors of Mathematics

Introduction

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizen's groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives.
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

Rationale for the Expanded Definition

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

Defining Basic Skills

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

Ten Basic Skill Areas

Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill.

Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, Interpreting, and Constructing Tables, Charts, and Graphs

Students should know how to read and draw conclusions from simple tables, maps, charts and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

Basic Skills and the Students Future

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.

Scope of Skill Development		Expected Outcomes
EXPANDED SKILLS Mathematical skills beyond those described here plus a desire to learn more.	➡	**POTENTIAL LEADERS** Employment and educational opportunities will continue to increase as mathematical skills continue to grow.
BASIC SKILLS The skills described here.	➡	**EMPLOYMENT VERY LIKELY** Employment opportunities are predictable. Doors to further education opportunities are open.

MINIMAL SKILLS
Limited skills, primarily computation. Little exposure to the other skill areas described here.

➡

LIMITED OPPORTUNITIES
Unemployment likely. Potential generally limited to low-level jobs.

Minimum Essentials For High-School Graduation

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

Developing the Basic Skills

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle. The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

Evaluating and Reporting Student Progress

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.

b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

Conclusion

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not back but **forward** to the essential needs of adults in the present and future.

You are encouraged to make and distribute copies of this paper.

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Appendix C

Suggested Curriculum Guide Format

Concept	Objectives	Skills	Suggested activities	Suggested resources	Methods for performance appraisal
	General objectives	Subject area			
	Enabling objectives	Thinking			
		Study			

Glossary for Teachers

GLOSSARY

Glossary for Teachers

The purpose of the glossary is for clarification of terms for the teacher and is not to be used by pupils. Each teacher should have a reputable student mathematics dictionary for use by the pupils.

Abscissa If the ordered pair of numbers (a,b) are the coordinates of a point of a graph, the number a is the abscissa. For comparison, see Ordinate.

Absolute Error One-half the smallest marked interval on the scale being used.

Absolute Value The absolute value of the real number a is denoted by a . If $a > 0$ then $a = a$ and if $a < 0$, $a = -a$. On the number line absolute value is the distance of a point from zero.

Accuracy The accuracy of a measurement depends upon the relative error. It is directly related to the number of significant digits in the measured quantity.

Additive Identity The number I in any set of numbers that has the following property: $I + a = a$ and $a + I = a$ for all a in the set. The symbol for the additive identity is usually 0; in the complex numbers it is $0 + 0i$, and in some systems bears no resemblance to zero.

Additive Inverse For any given number a in a set of numbers the inverse, usually designated by $(-a)$ is that number which when added to a will give the additive identity.

Algebraic Expression An algebraic expression may be a single numeral or a single variable; or it may consist of combinations of numbers and variables, together with symbols of operation and symbols of grouping.

Algorithm (Also Algorism) Any pattern of computational procedure.

Angle The set of all points on two rays which have the same end-point. The end-point is called the *vertex* of the angle, and the two rays are called the *sides* of the angle.

Angular Velocity The amount of rotation per unit of time.

Approximate Measure Any measure not found by counting.

Approximation An estimated answer. Also a decimal approximation is a method of finding any desired decimal representation of a number by placing it within successively smaller intervals.

Arc If A and B are two points of a circle with P as center, the arc AB is the set of points of the circle which are in the interior of $\angle APB$ and the two points, A and B, the points of intersection of the circle and the angle.

Area of a Surface Area measures the amount of surface.

Arithmetric Means The terms that should appear between two given terms so that all the terms will be an arithmetic sequence.

Arithmetic Sequence (Also Progression) A sequence of numbers in which there is a common difference between any two successive numbers.

Arithmetic Series The indicated sum of an arithmetic sequence.

Array A rectangular arrangement of elements in rows and columns.

Associative Property A basic mathematical concept that the way terms are grouped as certain types of operations are performed does not affect the result. The laws of addition and multiplication are stated as $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

Asymmetric Having no point, line or plane of symmetry.

Average A measure of central tendency. See mean.

Axis of Symmetry A line is called an axis of symmetry for a curve if it separates the curve into two portions so that every point of one portion is a mirror image in the line of a corresponding point in the other portion.

Base The first collection in a number series which is used as a special kind of one. It is used in combination with the smaller numbers to form the next number in the series. In the decimal system of numeration, eleven, which is one more than the base of 10, literally means 10 and one. Twenty means two tens or two of the base. In an expression such as a^n, *the quantity* a is called the *base* and n the exponent.

Base of a Numeration System The base of a numeration system is the number of units in a given digits place which must be taken to denote one in the next higher place.

Base Ten A system of numeration or a place-value arithmetic using the value 10 as its base value.

Basic Facts The name given to any operational table in a base of place-value arithmetic, as, basic addition tables, subtraction tables, multiplication tables, division tables, power tables, logarithmic tables. Basic addition facts include all addition facts in which neither of the addends exceeds 9. Basic subtraction facts include all the subtractions facts which correspond to all basic addition facts. The products formed by ordered pairs composed of the numbers 0 through 9 are called basic multiplication facts. Basic division facts include all the division facts which correspond to the basic multiplication facts such that $a/b = c$ provided $b \times c = a$, except $a/0$ is meaningless for all a.

Betweenness B is between A and C if A, B and C are distinct points on the same line and $AB + BC = AC$.

Bias When the method of selecting samples does not satisfy the condition that every possible sample that can be drawn has an equal chance of being selected, the sampling process is said to be biased.

Binary Operation An operation involving two numbers such as addition; similarly, a unary operation involves only one number as "the square of."

Binary Numeration System A system of notation with base two. It requires only two symbols, 0 and 1.

Bounded A point set S is bounded if and only if there is a circle (or sphere in 3-space) such that S lies entirely in the interior of that circle (sphere).

Cardinal Number If two sets can be put in one-to-one correspondence with each other, they are said to have the same cardinal number. A whole number which answers the question of how many in a given finite set is called the cardinal number of a set.

Cartesian Coordinates In a plane, the elements of ordered pairs which are distances from two intersecting lines or axes, which are usually perpendicular. The distances from one line is measured along a parallel to the other line.

Cartesian Plane A plane described by a Cartesian coordinate system.

Cartesian Product The Cartesian product of two sets A and B, written $A \times B$ and read "A cross B" is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$.

Central Tendency A measure of the trend of occurrences of an event. See mean, median and mode.

Check To verify the correctness of an answer or solution. It is not to be confused with *prove*.

Circle The set of points in a given plane each of which is at a given distance from a given point of the plane. The given point is called the *center*, and the given distance is called the *radius*.

Circumference The length of the closed curved line which is the circle.

Closed Curve (Simple) A path which starts at one point and comes back to this point without intersecting itself represents a *simple* closed curve.

Closure, Property of A set is said to have the property of closure for any given operation if the result of performing the operation on any two members of the set is a number which is also a member of the set.

Collection Elements or objects united from the viewpoint of a certain common property; as collection of pictures, collection of stamps, numbers, lines, persons, ideas.

Combination A combination of a set of objects is any subset of a given set. All possible combinations of the letters a, b and c are a, b, c, ab, ac, bc, abc. Also see permutation.

Commutative Property A basic mathematical concept that the order in which certain types of operations are performed does not affect the result. Addition is commutative, for example, $2 + 4 = 4 + 2$. Multiplication is commutative, for example, $2 \times 4 = 4 \times 2$.

Compass (or Compasses) A tool used to construct arcs and circles.

Complement Set The complement of a set S is denoted by S and contains all elements in the Universal Set that are not in the set S.

Complex Fraction A fraction that has one or more fractions in its numerator or denominator.

Complex Number Any number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

Composite Number A counting number which is divisible by a smaller counting number different from 1.

Congruent Two configurations which are such that when superimposed every point of either one lies on the other. Two figures have the same size and shape. Angles having the same measure are congruent.

Conjugate Complex Numbers The conjugate of the complex number $a + bi$ is the complex number $a - bi$.

Conjunction A statement consisting of two statements connected by the word *and*. An example is $x + y = 7$ *and* $x - y = 3$. The solution set for a conjunction of this type is the intersection of the solution sets of the separate statements.

Conditional Equation An equation that is true for only certain values of the variable, for example, $x + 3 = 7$.

Conic, Conic Section The curves which can be obtained as plane sections of a right circular cone.

Consistent System A system whose solution set contains at least one member.

Constant A particular member of a specified set.

Convergent Sequence A sequence that has a limit.

Coordinate Plane A plane whose points are named by ordered pairs of numbers which measure the distances from two intersecting lines. Each distance is measured from one line along a parallel to the other line.

Coterminal Angles Two angles which have the same initial and terminal sides but whose measures in degrees differ by 360 or a multiple of 360.

Countable In set theory, an infinite set is countable if it can be put into one-to-one correspondence with the natural numbers.

Counting Numbers The counting numbers are 1, 2, 3, 4, . . .

Decimal Fraction A linear array of integers that represent a fraction, every decimal place indicating a multiple of a positive or negative power of ten. For example, the decimal $.1 = \frac{1}{10}$, $.12 = \frac{12}{100}$, .003

Deductive Reasoning The process of using previously assumed or known statements to make an argument for new statements. The process of making inferences by reasoning from the general to the specific. For comparison, see Inductive Reasoning.

Degree In angular measure, a standard unit that is $\frac{1}{90}$ of the measure of a right angle. In arc measure, one of the 360 equal parts of a circle.

Degree of a Polynomial The general polynomial $a_0x^n + a_1 x^{n-1} + \ldots + a_{n-1}x^1 + a^n$ is said to be of degree n if $a_0 = 0$. For example, the degree of the polynomial $4x^3 + x - 5$ is 3.

Denominator The lower term in a fraction. It names the number of equal parts into which a number is to be divided. For example, the denominator of the fraction 3/4 is 4. For comparison see numerator.

Dependent Linear Equations Equations that have the same solution set.

Deviation The difference between the particular number and the average of the set of numbers under consideration is the deviation.

Difference The answer or result of a subtraction. For 8−5, the difference is 3.

Digit Any one of the ten symbols used in our numeration system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (from the Latin, "digitus," or "finger").

Dihedral Angle The set of all points of a line and two noncoplanar half-planes having the given line as a common edge. The line is called the *edge* of the dihedral angle. The *side* or *face* consists of the edge and either half-plane.

Direct Variation The number y varies directly as the number x if $y = kx$ where k is a constant.

Disc The union of a simple closed curve in a plane and its interior.

Disjunction A statement consisting of two statements connected by *or*, for example, $x + y = 7$ *or* $x - y = 3$. The solution set of this type of disjunction is the union of the solution sets of the separate statements.

Distributive Property Links addition and multiplication. Examples of the distributive property of multiplication over addition are as follows.

$$3\,(10 + 4) = (3 \times 10) + (3 \times 4)$$

$$4\,(3 + \tfrac{1}{2}) = (4 \times 3) + (4 \times \tfrac{1}{2})$$

$a\,(b + c) = ab + ac.$

Divergent Sequence A sequence that is not convergent.

Divisible An integer a is divisible by an integer b if and only if there is some integer c such that $b \times c = a$.

Division The inverse of multiplication. The process of finding how many times one quantity or number is contained in another. For any real numbers a and b, $b = 0$, $a \div b$ means a multiplied by the reciprocal of b. Also, $a \div b = c$, if and only if $a = bc$.

Domain of a Variable The set of all possible values of an independent variable of a function. Compare range.

Duodecimal Numeration System A system of notation with base twelve. It requires twelve symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E.

Edge The line segment of intersection of two faces of a polyhedron.

Elements In mathematics the individual objects included in a set.

Empty Set The set which has no elements. The symbol for this set is ϕ or $\{\ \}$.

End Point The point on a line from which a ray extends is called the end point of the ray. Similarly line segments have two endpoints.

Equality The relation *is equal to* denoted by the symbol "=."

Equation A sentence (usually expressed in symbols) in which the verb is "is equal to."

Equivalent Equations Equations that have the same solution set.

Equivalence Relation Any relation which is reflexive, symmetric and transitive, for example, reflexive: $a = a$; symmetric: if $a = b$ then $b = a$ and transitive: if $a = b$ and $b = c$, then $a = c$.

Equivalent Fractions Two fractions which represent the same number.

Equilateral Triangle A triangle whose sides have equal length.

Estimate A quick and frequently mental operation to ascertain the approximate value of an involved operation.

Even Number An integer that is divisible by 2. All even numbers can be written in the form $2n$, where n is an integer.

Expanded Exponential Form The expanded exponential form of a numeral is the form in which the additive, multiplicative and place value properties of a numeration system are explicitly indicated. The value of each place is written in exponential form, for example, $365 = 3(10^2) + 6(10^1) + 5(10^0)$.

Exponent In the expression a^n the number n is called an exponent. If n is a positive integer it indicates how many times a is used as a factor.

$a^n = a \times a \times \ldots \times a$

n factors

Under other conditions exponents can include zero, negative integers, rational and irrational numbers.

Exponential Equation An equation in which the independent variable appears as an exponent.

Exponential Function A function defined by the exponential equation $y = a^x$ where $a > 0$.

Extraneous Roots Those roots in the solution set of a derived equation which are not members of the solution set of the original equation.

Extrapolating Estimating the value of a function greater than or less than the known values; making inferences from data beyond the point strictly justified by the data.

Factor The integer m is a factor of the integer n if $m \times q = n$ where q is an integer. The polynomial $R(x)$ is a factor of the polynomial $P(x)$ if $R(x) \times Q(x) = P(x)$ where $Q(x)$ is a polynomial. *Factorization* is the process of finding the factors.

Factorial The expression "$n!$" is read n factorial. $n! = n(n-1)(n-2) \ldots 2 \times 1$.

Figurate Numbers Figurate numbers include the numbers more commonly referred to as square numbers, triangular numbers, etc.

Finite Set In set theory, a set which is not infinite. A set whose cardinal number is a whole number.

Fraction An indicated quotient of two quantities such as $\frac{a}{b}$ where a and b are numbers, with b not zero.

Frequency A collection of data is generally organized into several categories according to specified intervals or subcollections. A frequency is the number of scores or measures in a particular category.

Frequency, Cumulative The sum of frequencies preceding and including the frequency of measures in a particular category is the cumulative frequency.

Frequency Distribution A tabulation of the frequencies of scores or measures in each of the categories of data.

Function A relation in which no two of the ordered pairs have the same first member. Also, alternately, a function consists of (1) a set A called the domain, (2) a set B called the range, (3) a table, rule, formula or graph which associated each member of A with exactly one member of B.

Fundamental Theorem of Arithmetic Any positive integer greater than one may be factored into primes in essentially one way. The order of the primes may differ but the same primes will be present. Alternately, any integer except zero can be expressed as a unit times a product of its positive primes.

Fundamental Theorem of Fractions If the numerator and denominator are both multiplied (or divided) by the same non-zero number, the result is another name for the fraction.

Geometric Means The terms that should appear between two given terms so that all of the terms will form a geometric sequence.

Geometric Sequence A sequence in which the ratio of any term to its predecessor is the same for all terms.

Geometric Series The indicated sum of a geometric sequence.

Graph A pictorial representation of a set of points associated with a relation which involves one or more variables.

Greatest Integer Function Is defined by the rule $f(x)$ is the greatest integer not greater than x. It is usually denoted by the equation $f(x) = [x]$

Greatest Lower Bound A lower bound a of a set S of real numbers is the greatest lower bound of S if no lower bound of S is greater than a.

Hemisphere If a sphere is divided into two parts by a plane through its center, each half is called a hemisphere.

Histogram A bar graph representing a frequency distribution. The width of each of the contiguous rectangular bars represents the range of measures within a particular category, and the height of each of the bars represents the frequency of measures in the same category.

Identical Equation A statement of equality, usually denoted by $\equiv$ which is true for all values of the variables. The values of the variable which have no meaning are excluded, for example, $(x + y)^2 \equiv x^2 + 2xy + y^2$.

Identity A number, I, in a set of numbers that for a given operation, $*$, has the property that $I*a = a*I = a$ for all a in the set. See Additive Identity and Multiplicative Identity.

Inconsistent System of Equations A system whose solution set is the empty set.

Independent Events Two events are said to be independent if the occurrence of one does not affect the probability of occurrence of the other.

Independent System of Equations A system of equations that are not dependent.

Index The number used with a radical sign to indicate the root ($\sqrt[3]{\quad}$) In this example the index is three.) If no number is used, the index it two. ($\sqrt{\quad}$).

Inductive Reasoning The process of drawing a conclusion by observing what happens in a number of particular cases. Reasoning from the particular to the general. For comparison, see Deductive reasoning.

Inequality The relation in which the verb is one of the following—is not equal to, is greater than or is less than, denoted by the symbols $=$, $>$, $<$; respectively.

Infinite Decimal (Also non-terminating) A decimal representation that has an unending string of digits, other than *zeroes*, to the right of the decimal point.

Infinite Repeating Decimal A decimal representation containing a finite block of digits which repeats endlessly.

Infinite Set In set theory, a set which can be placed in one-to-one correspondence with a proper subset of itself.

Integer Any one of the set of numbers which consists of the natural numbers, their opposites and zero.

Intercept If the points whose coordinates are $(a,0)$ and $(0,b)$ are points on the graph of an equation, they are called intercepts. The point whose coordinators are $(a,0)$ is the x-intercept, and the point whose coordinates are $(0,b)$ is the y-intercept.

Interpolation The process of estimating a value of a function between two known values other than by the rule of the table of the function.

Intersecting Lines Two or more lines that pass through a single point in space.

Intersection of Sets If A and B are sets, the intersection of A and B, denoted by $A \cap B$, is the set of all elements which are members of *both A and B*.

Inverse of an Operation That operation which, when performed after a given operation, annuls the given operation. Subtraction of a quantity is the inverse of addition of that quantity. Addition is likewise the inverse of subtraction. Multiplication and division are inverse operations.

Inverse Variation The number y is said to vary inversely as the number x if $x \times y = k$ where k is a constant.

Irrational Number An irrational number is not a rational number. That is, it is a number that cannot be expressed in the form $\frac{a}{b}$ where a and b are integers. The union of the set of rationals and the set of irrationals is the set of real numbers.

Joint Variation A quantity varies jointly as two other quantities if the first is equal to the product of a constant and the other two, for example, y varies jointly as x and w if $y = kxw$.

Lattice Points An array of points named by ordered pairs.

Least Common Multiple The least common multiple of two or more numbers is the common multiple which is a factor of all the other common multiples.

Least Upper Bound An upper bound b of a set S of real numbers is the least upper bound of S if no upper bound of S is less than b.

Linear Equation An equation in standard form in which the sum of the exponents of the variable in any term equals one. For example $y = 3x-2$ is a linear equation. The graph of a linear equation is a line.

Linear Measure A measure used to determine length.

Logarithm The exponent that satisfies the equation $b^x = n$ is called the logarithm of n to the base b for any given positive number n. For example, $\log_{10} 1000 = 3$ since $10^3 = 1000$.

Lower Bound A number a is called a lower bound of set S of real numbers if $a<x$ for every $x \in S$.

Magic Square A square of numbers possessing the particular property that the sums in each row, column and diagonal are the same.

Matrix A rectangular array of numbers.

Example

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Mean In a frequency distribution, the sum of the n measures divided by n is called the mean. The mean is commonly called the average.

Measurement Measurement of a quantity implies that a number is assigned to represent its magnitude. Usually the assignment can be made by a simple comparison. The magnitude of the quantity is compared to a "standard" quantity, the magnitude of which is arbitrarily chosen to have the measure 1.

Median In a frequency distribution, the measure that is in the middle when elements are arranged from highest to lowest is called the median. In geometry, a median of a triangle is a line joining a vertex to the midpoint of the opposite side.

Mode In a frequency distribution, the measure which appears most frequently in the set of observations is called the mode. There may be more than one mode in a set of measures.

Modulo Arithmetic For a given positive integer n, *modulo* n is obtained by using the integers $0 \ldots n-1$ and defining addition and multiplication by letting the sum of $a + b$ and the product of $a\,b$ be the remainder after division by n of the ordinary sum and product of a and b. (This is often called clock arithmetic.)

Modulus A statement of the type x is congruent to y modulus (or modulo) w, w is the modulus of the congruency. If 2 is congruent to 9, then the modulus is 7.

Multiple If a and b are integers such that $a = b \times c$ where c is an integer, then a is said to be a multiple of b.

Multiplication A short method of adding like groups or addends of equal size. It may be illustrated on a number line by counting forward by equal groups.

Multiplicative Identity The number I in any set of numbers that has the following property: $I \times a = a \times I = a$ for all a in the set. The symbol for the multiplicative identity is usually 1.

Multiplicative Inverse The multiplicative inverse of a non-zero number a is the number b such that $a \times b = 1$. It is usually designated by $\frac{1}{a}$ or a^{-1}.

Mutually Disjoint Sets Two sets having no elements in common.

Mutually Exclusive Events Events which cannot occur simultaneously. Mutually exclusive subsets are subsets that are disjoint.

Natural Numbers Any of the set of counting numbers. The set of natural numbers is an infinite set; it has a smallest member (1) but no largest member: 1, 2, 3, 4, ...

Null Set A set containing no elements. It is sometimes called an empty set. The symbol for the null set is ϕ or $\{\,\}$.

Number System A number system consists of a set of numbers, two operations defined on the set, the properties belonging to the set and a definition for equivalence between any two members of the set.

Numeration System A coding system for recording numerals. Modern systems of numeration are characterized by a set of symbols, or digits, a place value scheme and a base.

Numerator The upper term in a fraction. For example, the numerator of the fraction 3/4 is 3. For comparison see denominator.

Numeral A written symbol for a number, for example, several numerals for the same number are 8, VIII, $7 + 1$, $10 - 2$, $\frac{16}{2}$.

Obtuse Angle If the degree measure of an angle is between 90 and 180, the angle is called an obtuse angle.

Odd Number An odd number is an integer that is not divisible by 2; any number of the form $2n + 1$, where n is an integer.

One-To-One Correspondence A pairing of the members of a set A with members of a second set B such that each member of A is paired with exactly one member of B, and each member of B is paired with exactly one member of A.

Open Sentence An open sentence is a sentence involving one or more variables, and the question of whether it is true cannot be decided until definite values are given to the variables, for example, $x + 5 = 7$.

Ordered N-Tuple A linear array of numbers $(a_1, a_2, a_3, \ldots, a_n)$ such that a_1 is the first number, a_2 is the second number, a_3 is the third number, . . . and a_n is the nth number.

Ordered Pair A pair of numbers (a,b) where a is the first member and b is the second member of the pair.

Ordinal Number A number that denotes order of the members in a set.

Ordinate If an ordered pair of numbers (a,b) are coordinates of a point P,b is called the ordinate of P. For comparison, see Abscissa.

Parallel Lines Two straight lines in a plane that do not intersect however far extended.

Parallelogram A quadrilateral whose opposite sides are parallel.

Parameter An arbitrary constant or a variable in a mathematical expression, which distinguishes various specific cases.

Partial Product Used in elementary arithmetic with regard to the written algorithm of multiplication. Each digit in the multiplier produces one partial product. The final product is then the sum of the partial products.

Partial Quotient In long division, any of the trial quotients that must be added to obtain the complete quotient.

Perimeter The sum of the measures of the sides of a polygon. The measure of the outer boundary of a polygon region.

Period The number of digits set off by a comma in an integer or the integral part of a mixed decimal. In a repeating decimal the periods is the sequence of digits that repeats.

Periodic Function A function from R to R, where R is the set of real numbers, is called periodic if, and only if, $f(x)$ is not the same for all x and there is a real number p such that $f(x + P) = f(x)$ for all x in the domain of f. The smallest positive number p for which this holds is called the period of the function.

Permutation Permutation is an ordered arrangment of all or part of the members in a set. All possible permutations of the letters, a, b and c are a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, cba.

Perpendicular Lines Two lines intersecting to form a pair of angles of equal measure are said to be perpendicular.

Place Value The value of a numeral is dependent upon its position. In the number 324, for example, each digit has a place value 10 times that of the place value of the digit to its immediate right.

Plane Angle Through any point on the edge of a dihedral angle pass a plane perpendicular to the edge intersecting each side in a ray. The angle formed by these rays is called the plane angle of the dihedral angle.

Point Set A collection of points such as the set of points on a line segment or within a circle.

Polar Coordinates An ordered pair used to represent a complex number. The first member of the pair is the number of units in the radius vector, and the second member is the angle of rotation of the radius vector.

Polygon A simple closed curve which is the union of line segments is called a polygon.

Polyhedron A solid bounded by plane polygons. The bounding polygons are the *faces*, the intersections of the faces are the *edges* and the points where three or more edges intersect are the *vertices*.

Polynomial An algebraic expression of the form $a_o x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$ sometimes designated by the symbol $P(x)$.

Polynomial Equation A statement that $P(x) = 0$.

Polynomial Function A function defined by a polynomial equation or $f: x \rightarrow P(X)$.

Precision The precision of a measurement is inversely related to the absolute error. Thus the smaller the absolute error, the greater the precision.

Prime Number A counting number other than one, which is divisible only by itself and one.

Prism If a polyhedron has two faces parallel and its other faces in the form of parallelograms, it is called a prism.

Probability The numerical of the likelihood of an event is called the probability of the event. It is a rational number p such that $0 \geqslant p \geqslant 1$.

Proper Subset A subset R is a proper subset of a set S if R is a subset of S and $R \neq S$. R is a proper subset of S is indicated by $R \subset S$. See Subset.

Pyramid A polyhedron, one of whose faces is a polygon of any number of sides and whose other faces are triangles having a common vertex.

Quadrilateral A polygon formed by the union of 4 line segments.

Quinary System of Numeration A system of notation with the base 5. It requires only five symbols or digits—0, 1, 2, 3, 4.

Radian Measure Angular measure where the unit is the measure of an angle whose arc on a circle with center at vertex of angle is equal in length to the radius of the circle.

Radius Any line segment with one endpoint at the center of a circle and the other endpoint on the circle is called a radius of the circle.

Radius Vector A line segment with one end fixed at the origin on the Cartesian plane and rotating from an initial position along the positive x-axis so that its free end point generates a circle.

Range (Statistics) The range of the set of numbers is the difference between the largest and smallest numbers in a set. For example, if the set of numbers vary from 7 to 25, the range is 18.

Range (of a Function) The set of all elements assigned to the elements of the domain by the rule of the function.

Rate Pair An ordered pair of counting numbers which expressed a rate relation — e.g., a rate of exchange. In general, a rate pair $\frac{a}{b}$, where a and b are counting numbers, expresses a ratio of the number of elements in one set to the number of elements in a second set.

Ratio The relative size of two quantities expressed as the quotient of one divided by the other: the ratio of 5 to 3 is written 5:3 or $\frac{5}{3}$.

Rational Expression A rational expression is a quotient of two polynomials or in symbols $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

Rational Number If a and b are whole numbers with b not zero, the number represented by the fraction $\frac{a}{b}$ is called a rational number.

Rational Numbers of Arithmetic In the elementary school, one generally defines a set of equivalent fractions to be a rational number. Alternatively, a rational number is an equivalence class of ordered pairs of integers a and $b, b \neq 0$.

Ray Let A and B be points on a line.—Then ray $\overrightarrow{AB}$ is the set which is the union of the segment $\overline{AB}$ and the set of all points C for which it is true that B is between A and C. The point A is called the *end point* of $\overrightarrow{AB}$.

Reciprocal Multiplicative inverse. For example, the reciprocal of 2/3 is 3/2 since $2/3 \times 3/2 = 1$.

Reciprocal Function Pairs of functions in the set of real numbers whose product is 1, for example, $(\sin \phi)(\csc \phi) = 1$.

Rectangle A parallelogram with right angles.

Reference Triangle For any angle on the Cartesian plane with vertex at the origin, the triangle formed by the radius vector, its projection on the x-axis and a line drawn from the end of the radius vector perpendicular to the x-axis is called the reference triangle.

Reflection in a Line A point P has a mirror image P' in the line $\overleftrightarrow{AB}$ if P, P' and $\overleftrightarrow{AB}$ all lie in the same plane with P and P' on opposite sides of $\overleftrightarrow{AB}$ and if the perpendicular distances PO and $P'O$ to the point O in $\overleftrightarrow{AB}$ are equal.

Reflexive Property If a is any element of a set and if R is a relation on the set such that aRa for all a, then R is reflexive.

Region The union of a simple closed curve and its interior.

Related Angle For any angle on the Cartesian plane, the related angle is the angle in the reference triangle formed by the radius vector and x-axis.

Relation A relation from set A to set B (where A and B may represent the same set or different sets) is any set of ordered pairs (a,b) such that a is a member of A and b is a member of B.

Relative Error Ratio of the absolute error to the measured value.

Relative Frequency The relative frequency is the number of measures in a given category divided by the total number of measures in all categories.

Relative Prime Two integers are relatively prime if they have no common factors other than $+1$ or -1; two polynominals are relatively prime if they have no common factors except constants.

Repeating Decimal A decimal fraction which never ends *and which* repeats a sequence of digits other than all zeros. It is indicated in this manner $-0.333 \ldots$ or $0.\overline{142857}$.

Restricted Domain Domain of a function or relation from which certain numbers are excluded for reasons such as division by zero is not permitted and need for the inverse of a function to be a function.

Right Angle Any of the four angles obtained at the point of intersection of two perpendicular lines. The angle made by two perpendicular rays. Its measure is 90 degrees.

Right Triangle A triangle with one right angle.

Rounding Off Replacing digits with zero's to a certain designated place in a number with the last remaining digit being increased or decreased under certain specified conditions.

Sample Space The set of all possible outcomes of an experiment.

Scalar In physical science, a quantity having magnitude but no direction. In a study of mathematical vector, any real number.

Scale A system of marks in a given order and at fixed intervals. Scales are used on rulers, thermometers and other measuring instruments and devices as an aid in measuring quantities.

Scientific Notation A notation generally used for very large or very small numbers in which each numeral is changed to the form $a \times 10^k$ where a is real number such that $1 \leqslant a < 10$ and k is any integer.

Example:

$$6{,}710{,}000 = 6.71 \times 10^6$$
$$.000000052 = 5.2 \times 10^{-8}$$

Segment For any two points A and B, the set of points consisting of A and B and all points between A and B is the line segment determined by A and B. The segment is a geometrical figure while the distance is a number which tells how far A is from B.

Sequence An ordered arrangement of numbers.

Series The indicated sum of a sequence.

Set A collection of particular things, as a set of numbers between 3 and 5, the set of points on the segment of a line or within a circle.

Set Builder Notation To describe the members of a very large or infinite set, it is often helpful to denote the set and its members as in this example—$\{x \; x \in R \text{ and } 0 \leq x \leq 1\}$, read "The set of all x such that x is a member of the set R of real numbers and x is greater than or equal to 0 and less than or equal to 1." The symbol device, $\{x \mid x \ldots\}$,read "the set of all x such that x..." is called set builder notation.

Significant Digit Any digit in a numeral not used solely for placement of the decimal point, for example, $\dot{7}\dot{0}\dot{3},000$· $.00\dot{5}\dot{6}$, $\dot{5}.\dot{0}\dot{0}$.

Similar Two geometric figures are similar if one can be made congruent to the other by using a transformation of similitude if one is a magnification or reduction of the other. Geometric figures are similar if their corresponding angles are congruent and corresponding line segments proportional.

Skew Lines Two lines which are not coplanar are said to be skew.

Slope The slope of a given segment (P_1P_2) is the number m such that $m = \frac{y_2 - y_1}{x_2 - x_1}$ where P_1 is the ordered pair (x_1, y_1) and P_2 is the ordered pair (x_2, y_2).

Solid Any simple closed surface; the term is usually used with reference to polyhedra (rectangular solids, pyramids), cylinders, cones and spheres.

Solution Set The set of values that satisfy (or make true) an equation or a system of equations.

Sphere The set of all points in space each of which is at a given distance from a given point. The given point is called the *center* of the sphere and the given distance is called the *radius*.

Square A quadilateral formed by four line segments of equal length which meet at right angles. A rectangle with sides of equal length.

Standard Deviation The square root of the arithmetic mean of the squares of the deviations from the mean.

Statistic An estimate of a parameter obtained from a sample, as of the population mean or standard deviation.

Statistics The concepts, measures and techniques related to methods of obtaining, organizing and analyzing data is included in statistics.

Subset A set contained within a set; a set whose members are members of another set. The fact that R is a subset of S is indicated by $R \subseteq S$. See also Proper Subset.

Subtraction To subtract the real number b from the real number a, add the opposite (additive inverse of b) to a. $a - b = a + (-b)$. Also, $a - b = c$ if and only if $a = b + c$.

Successor The successor of the integer a is the integer $a + 1$.

Summation Notation The symbol $\sum_{k=i}^{n} a_k$. The symbol Σ, the Greek letter "sigma," corresponds to the first letter of the word "sum" and is used to indicate the summing process. The n and i represent the upper and lower indexes and indicate that the summing begins with the ith term and includes the nth term, for example,

$$\sum_{k=2}^{5} a_k = a_2 + a_3 + a_4 + a_5.$$

When the summation includes infinitely many terms it is written $\sum_{k}^{\infty} a_k$. In this case there is no last term n because ∞ is not a number. The symbol ∞ is used to indicate that the summation is infinite.

Symmetric Property If a and b are any elements of a set and if R is a relation on the set such that aRb implies bRa, then the relation is said to have the symmetric property.

Term In a phrase which has the form of an indicated sum, $A + B$, A and B are called *terms* of the phrase.

Terminating Decimal (Also finite decimal) A decimal representation that contains a finite number of digits or that repeats an infinite sequence of zeros.

Topology A branch of mathematics which is the study of properties of point sets which are preserved under specific transformations.

Transitive Property If a, b and c are any elements of a set and if R is a relation on the set such that aRb and bRc imply aRc, then the relation is said to have the transitive property.

Trapezoid A quadilateral with two parallel sides. It is sometimes required that the other sides not be parallel.

Triangle If A, B and C are three non-collinear points in a given plane, the set of all points in the segments having A, B, C as their end points is called a triangle.

Unary Operation An operation involving one number such as "the cube of" or "the square root of."

Unbounded Not bounded.

Unequal Not equal, symbolized by $\neq$.

Union of Sets If A and B are two sets, the union of A and B is the set $A \cup B$ which contains all the elements and only those elements that are in A *or* in B, for example, $A = \{2,8,3\}$, $B = \{5,2,7,6\}$ then $A \cup B = \{2,8,3,5,7,6\}$.

Unique One and only one.

Universal Set The largest set under consideration in the context of a problem situation.

Upper Bound A number b is called an upper bound of a set S of real numbers if $b \geq x$ for every $x \in S$.

Variable A letter used to denote any one of a given set of numbers. Another name for variable is placeholder in an equation, for example, $x + 5 = 7$.

Vector In physical science, a quantity having magnitude and direction. In mathematics a vector is a matrix of one row or one column as $(a_1\ b_1\ c_1)$ or

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Vertex The point of intersection of two or more rays or line segments. A vertex in 3-space is formed such as for a polyhedron where 3 or more edges intersect. The plural form of vertex is vertices.

Volume The amount of space occupied by a solid or enclosed within it.

Whole Numbers The whole numbers are 0,1,2,3,4, ...

Mathematical Symbols

$\neq$	is not equal to
$\approx$	is approximately equal to
$>$	is greater than
$\ngtr$	is not greater than
$<$	is less than
$\nless$	is not less than
$\geq$	is greater than or equal to
$\ngeq$	is not greater than or equal to
$\leq$	is less than or equal to
$\nleq$	is not less than or equal to
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\cong$	is congruent to
$\sim$	is similar to
$\in$	is an element of
$\notin$	is not an element of
$\mathcal{U}$	universal set
S	solution set
$\overline{S}$	complement set
$A \times B$	Cartesian product set of sets A and B
a^{-n}	is interpreted as $\frac{1}{a^n}$ where $a \neq 0$
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\overleftrightarrow{AB}$	straight line containing points A and B
$\overline{AB}$	straight line segment with end points A and B.
$\overrightarrow{AB}$	ray from point A through point B
(a, b)	ordered pair a and b
$\{a\}$	set containing elements a

$\square, \Delta$	frames, place holders or nonspecified elements
$\phi \{\ \}$	the empty set
$\triangle ABC$	triangle with vertices A, B, and C applies to any polygon
$\angle ABC$	angle with point B as vertex
$\{\square \mid \square > 5\}$	the set of all $\square$ in the universal set such that $\square$ is greater than 5
$a:b$	ratio of a to b *(also written* $\frac{a}{b}$)
$\cup$	union of two sets
$\cap$	intersection of two sets

Careers in Mathematics

CAREERS IN MATHEMATICS

Careers in Mathematics

To the middle grades teacher

The purpose of this career education material is to provide suggestions to help students become aware of mathematics in various careers.

Suggested activities

The emphasis in career education in the elementary school is on awareness and exploration, **not** on career decision making. Career awareness should be approached through multi-disciplinary studies, for instance, through the study of community helpers. At the same time the service occupations are studied, other types of occupations should also be considered. The brochure enclosed here may be reproduced and used to help stimulate career awareness and exploration.

Some plans for its use include these.

1) Reproduce at least one copy for each child.

2) Let students circle those occupations that represent people they know.

3) Encourage your students to write additional occupations that are not included.

4) Have your students talk with a few people in these various occupations. Help them pose some questions they might ask. Here are some samples.
   a) What do you do?
   b) What do you like about your work?
   c) What don't you like about your work?
   d) What do I need to learn (especially in mathematics) to be able to do this work?
   e) When do you work?
   f) Where do you do your work?
   g) Would you come talk to our class about your work?

5) Work in small groups (with individuals) discussing, "What kinds of things I like to do" or, "How I feel about these jobs."

6) Let the student use the same sheet to identify or state different careers that they might be interested in pursuing.

7) Encourage them to use available resources (e.g., from the media center—books, films, film strips; local public library, people in the community; material in counselor's office) for a presentation (oral or written).

8) Let them play "Twenty Questions" or "What's My Line" with students playing roles in various occupations.

WILL MATHEMATICS BE A PART OF MY FUTURE?

Yes. Our society is using mathematics more and more. More doors are closed without it.

•

WHAT IS THE BEST WAY TO PREPARE FOR MY FUTURE WHILE I AM IN MIDDLE GRADES AND HIGH SCHOOL?

You can expect mathematics to be needed for your future work in schools and jobs. You need to be ready. Study mathematics now and in highschool to increase the number of careers available to you.

•

DO I NEED MATHEMATICS IN HIGH SCHOOL IF I PLAN TO ATTEND VOCATIONAL SCHOOL?

Yes, you should elect as many mathematics courses as you can schedule.

•

DO I NEED MATHEMATICS TO PREPARE FOR COLLEGE?

Yes, the college program you plan will determine the mathematics you need in high school.

CAN I GET A JOB AFTER HIGH SCHOOL WITHOUT TAKING MATHEMATICS?

Yes, but the better you can do mathematics you may have more chances for jobs and for advancement.

•

CAN I EXPECT MY EARNINGS TO INCREASE AS A RESULT OF STUDYING MORE AND MORE MATHEMATICS?

While money is important when selecting a career, you should include

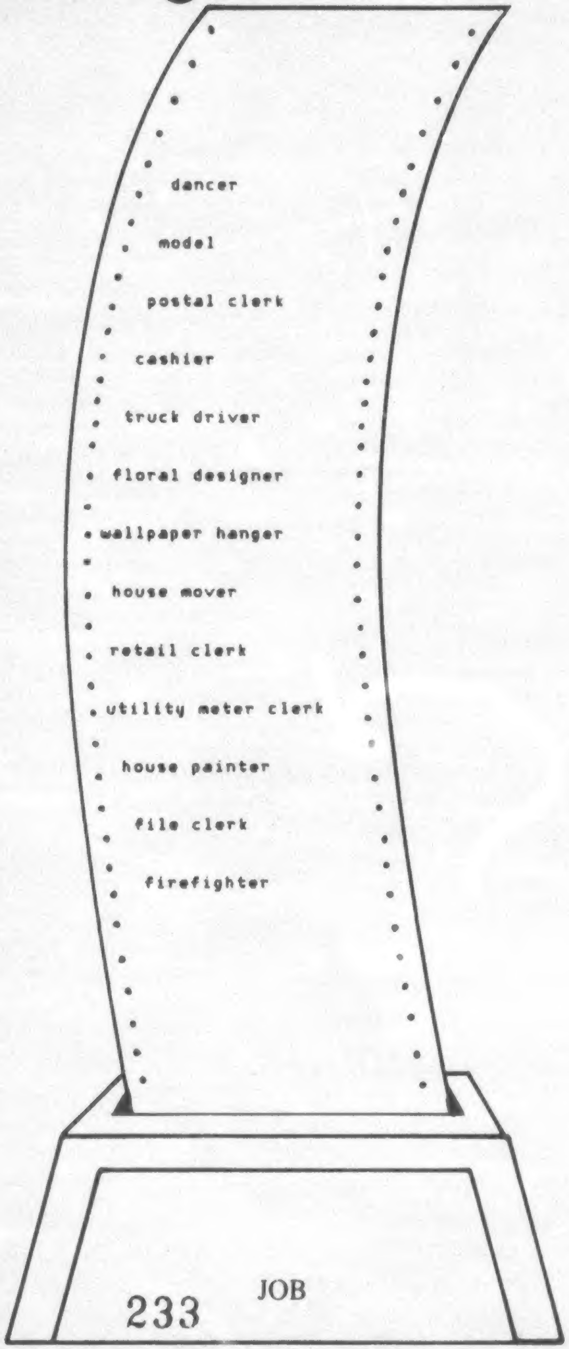
- job satisfaction
- job security
- opportunities for advancement
- employee benefits

•

HOW CAN I GET MORE INFORMATION?

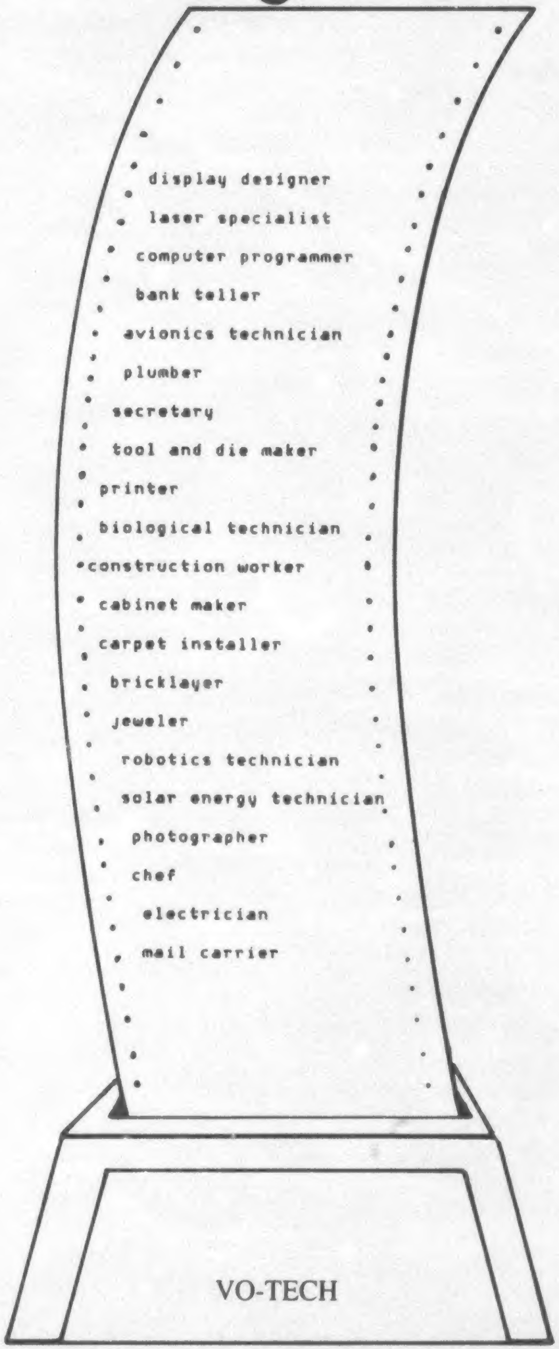
- Mathematics teachers
- Sources in the guidance office, such as the current edition of Occupational Outlook Handbook
- Sources in the media center
- Sources in the career education center

Mathematics for Career Plans



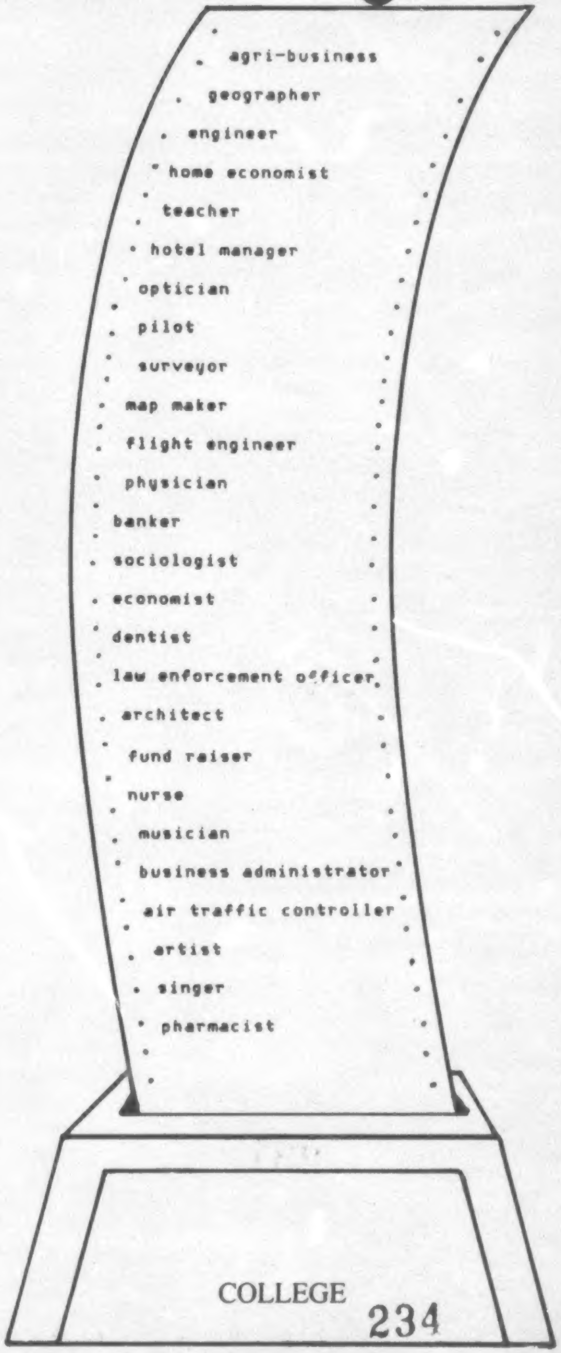
JOB

- dancer
- model
- postal clerk
- cashier
- truck driver
- floral designer
- wallpaper hanger
- house mover
- retail clerk
- utility meter clerk
- house painter
- file clerk
- firefighter



VO-TECH

- display designer
- laser specialist
- computer programmer
- bank teller
- avionics technician
- plumber
- secretary
- tool and die maker
- printer
- biological technician
- construction worker
- cabinet maker
- carpet installer
- bricklayer
- jeweler
- robotics technician
- solar energy technician
- photographer
- chef
- electrician
- mail carrier



COLLEGE

- agri-business
- geographer
- engineer
- home economist
- teacher
- hotel manager
- optician
- pilot
- surveyor
- map maker
- flight engineer
- physician
- banker
- sociologist
- economist
- dentist
- law enforcement officer
- architect
- fund raiser
- nurse
- musician
- business administrator
- air traffic controller
- artist
- singer
- pharmacist

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Title VI—Peyton Williams, Jr., Associate Superintendent of State Schools and Special Services
Title IX—Myra Tolbert, Coordinator
Section 504—Jane Lee, Coordinator of Special Education

Inquiries concerning the application of Title II, Title VI, Title IX or Section 504 to the policies and practices of the department may be addressed to the persons listed above at the Georgia Department of Education, Twin Towers East, Atlanta 30334; to the Regional Office for Civil Rights, Atlanta 30323; or to the Director, Office for Civil Rights, Education Department, Washington, D.C. 20201.

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Office of Instructional Services
Georgia Department of Education
Atlanta, Georgia 30334
Charles McDaniel
State Superintendent of Schools
1982

DOAS

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